Is polynomial interpolation in the monomial basis unstable?

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Definition

Given a function $F : [-1, 1] \to \mathbb{C}$, the Nth degree interpolating polynomial P_N of F satisfies $P_N(x_j) = F(x_j)$, for a set of $(N+1)$ distinct collocation points $\{x_j\}_{j=0,1,\ldots,N}$.

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The choice of collocation points matters. In this talk, we only consider collocation points with a small Lebesgue constant (e.g., Chebyshev points).

Polynomial interpolation in finite precision

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The standard choices:

- Lagrange polynomials.
- Orthogonal polynomials (Chebyshev, Legendre, etc).

Polynomial interpolation in the monomial basis

What about expressing P_N in the monomial basis?

$$P_N(x) = \sum_{k=0}^{N} a_k x^k$$

The previous linear system becomes

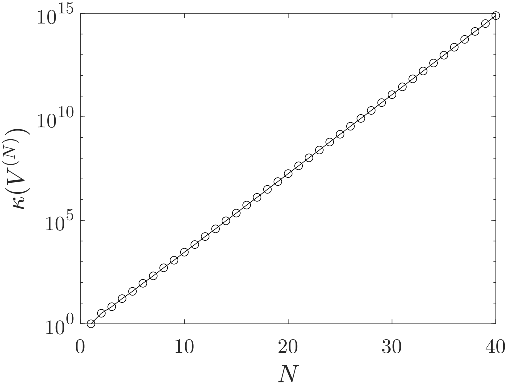
$$\underbrace{\begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^N \\ 1 & x_1 & x_1^2 & \cdots & x_1^N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_N & x_N^2 & \cdots & x_N^N \end{pmatrix}}_{V^{(N)}} \underbrace{\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_N \end{pmatrix}}_{a^{(N)}} = \underbrace{\begin{pmatrix} F(x_0) \\ F(x_1) \\ \vdots \\ F(x_N) \end{pmatrix}}_{f^{(N)}}.$$

$V^{(N)}$ is known as a Vandermonde matrix.

Monomial basis is ill-conditioned

Given any set of real collocation points, $\kappa(V^{(N)})$ grows at least exponentially fast.

Example: when the Chebyshev points are used for collocation:



Numerical experiments

Let's run some experiments. The following quantities will be reported.

- $\|F - \widehat{P}_N\|_{L^\infty([-1,1])}$: Monomial approximation error.
  Denoted by the label "monomial".
- $\|F - P_N\|_{L^\infty([-1,1])}$: Exact polynomial interpolation error, estimated using the Barycentric interpolation formula.
  Denoted by the label "Lagrange".

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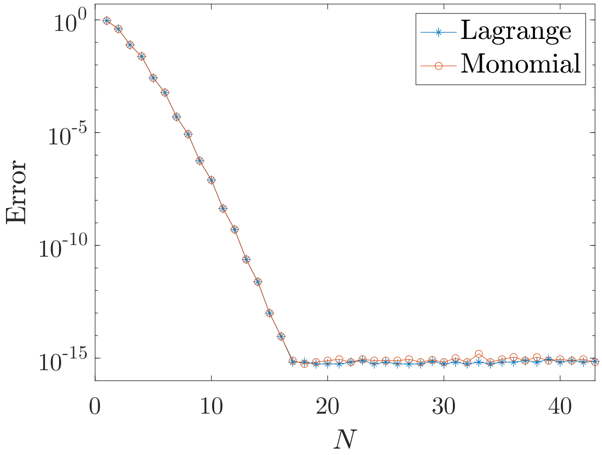
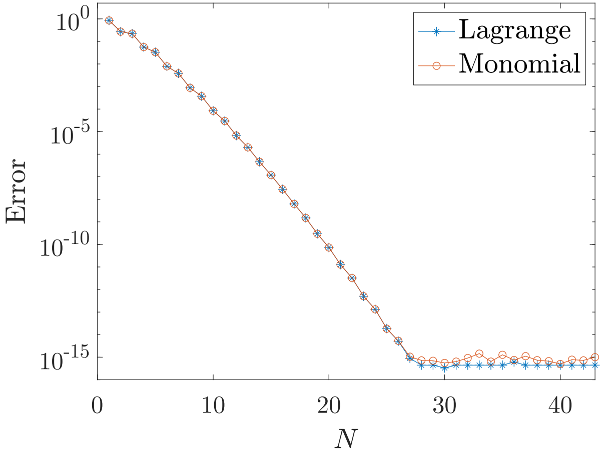


Numerical experiments

$$F(x) = e^{-2(x+0.1)^2}$$



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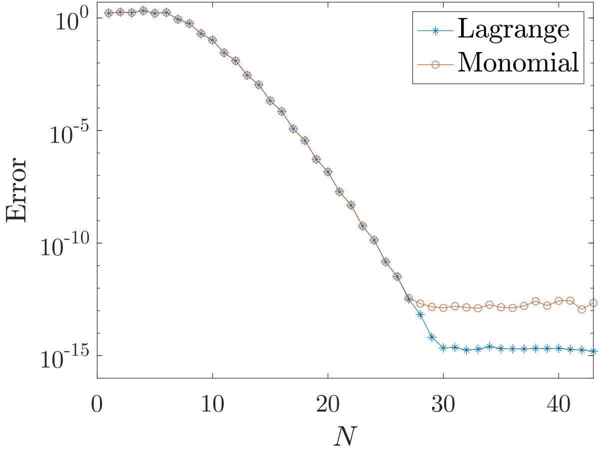
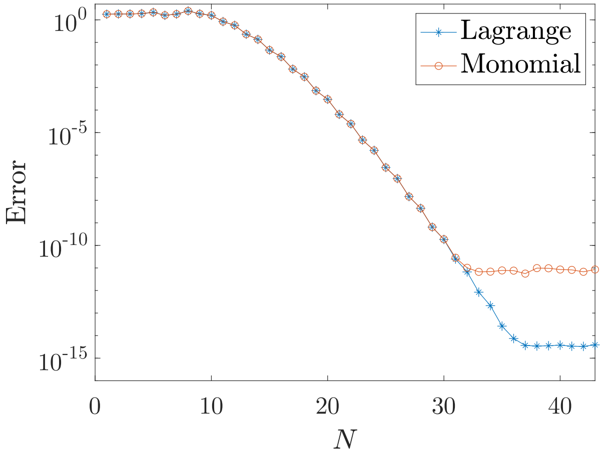


Numerical experiments

$$F(x) = \cos(12x + 1)$$



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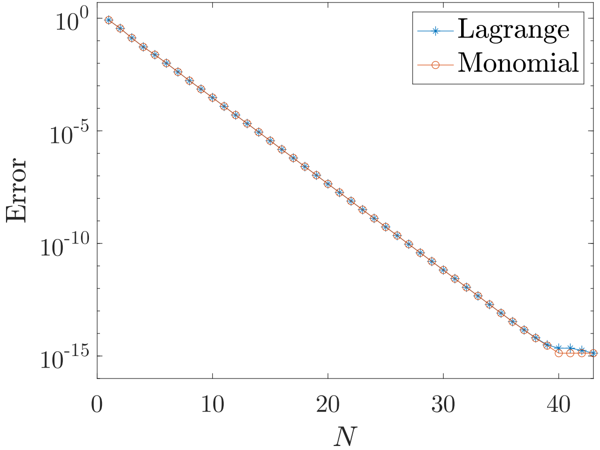

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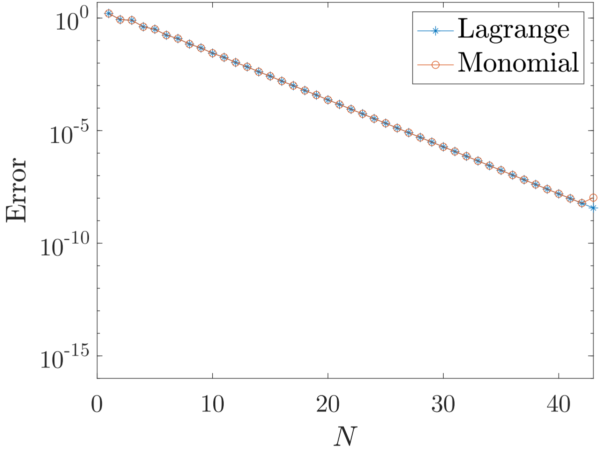

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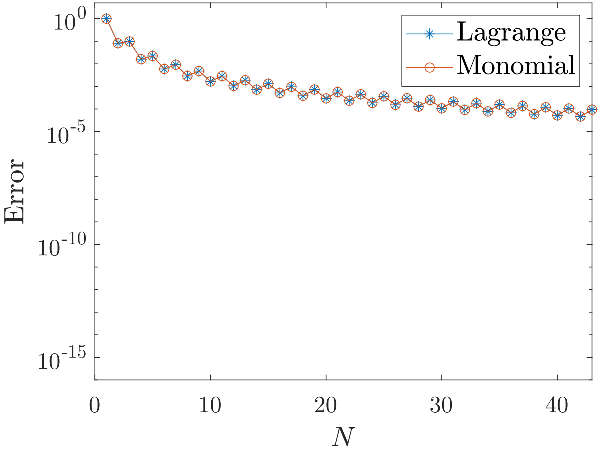

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Before I explain why, I'll present an application.

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On the other hand, the monomial basis works for any domain, is extremely handy, and is much cheaper to evaluate.

Rethinking interpolation

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$$\Longrightarrow$$

extremely inaccurate monomial coefficients

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What's really important is the backward error, i.e.,

$$\| V^{(N)} \widehat{a}^{(N)} - f^{(N)} \|_2,$$

of the numerical solution $\widehat{a}^{(N)}$ to the Vandermonde system $V^{(N)} a^{(N)} = f^{(N)}$.

Rethinking interpolation

The difference between the exact interpolating polynomial P_N and the computed monomial expansion $\widehat{P}_N$ satisfies

$$\|P_N - \widehat{P}_N\|_{L^\infty(\Gamma)} \leq \Lambda_N \|V^{(N)} \widehat{a}^{(N)} - f^{(N)}\|_2.$$

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How large will the backward error be?

Backward stable linear system solver

When a backward stable linear system solver is used to solve the Vandermonde system $V^{(N)}a^{(N)} = f^{(N)}$, the numerical solution $\widehat{a}^{(N)}$ is the exact solution to

$$\left(V^{(N)} + \delta V^{(N)}\right)\widehat{a}^{(N)} = f^{(N)},$$

for some $\delta V^{(N)} \in \mathbb{C}^{(N+1)\times(N+1)}$ that satisfies

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where u denotes machine epsilon and $\gamma_N = \mathcal{O}(\|V^{(N)}\|_2)$.

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It follows that

$$\|V^{(N)}\widehat{a}^{(N)} - f^{(N)}\|_2 = \|\delta V^{(N)}\widehat{a}^{(N)}\|_2 \leq u \cdot \gamma_N \|\widehat{a}^{(N)}\|_2.$$

A priori estimate

Lemma

If $\|(V^{(N)})^{-1}\|_2 \leq \frac{1}{2u\cdot\gamma_N}$, *then* $\frac{2}{3}\|a^{(N)}\|_2 \leq \|\widehat{a}^{(N)}\|_2 \leq 2\|a^{(N)}\|_2$.

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Corollary (Finite-precision interpolation error)

If $\|(V^{(N)})^{-1}\|_2 \leq \frac{1}{2u\cdot\gamma_N}$, *then*

$$\|F - \widehat{P}_N\|_{L^\infty(\Gamma)} \leq \|F - P_N\|_{L^\infty(\Gamma)} + 2u\cdot\gamma_N\Lambda_N\|a^{(N)}\|_2.$$

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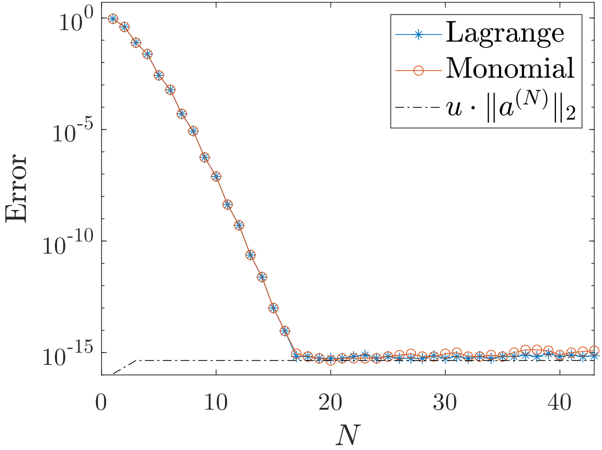
$$F(x) = \cos(2x + 1)$$



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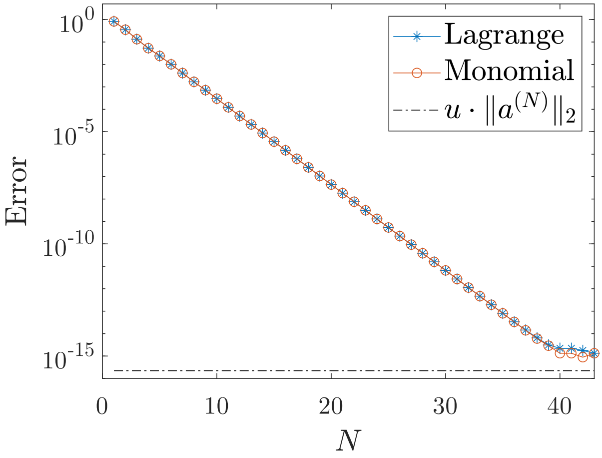

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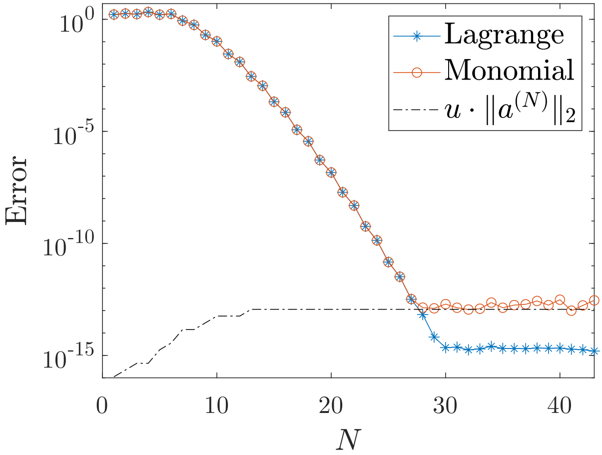

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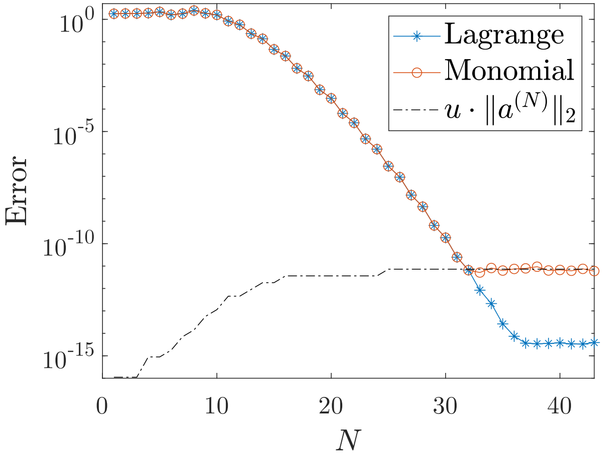

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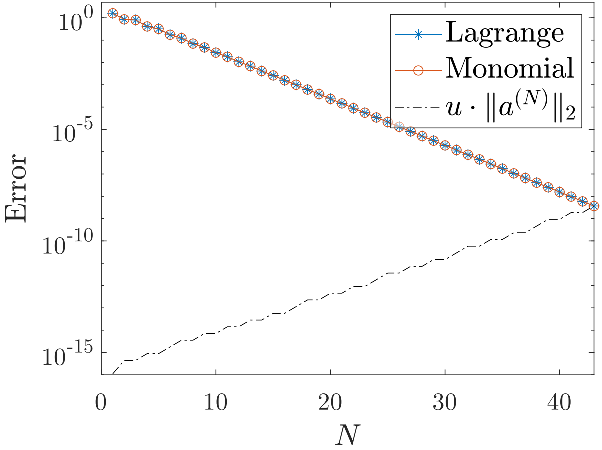

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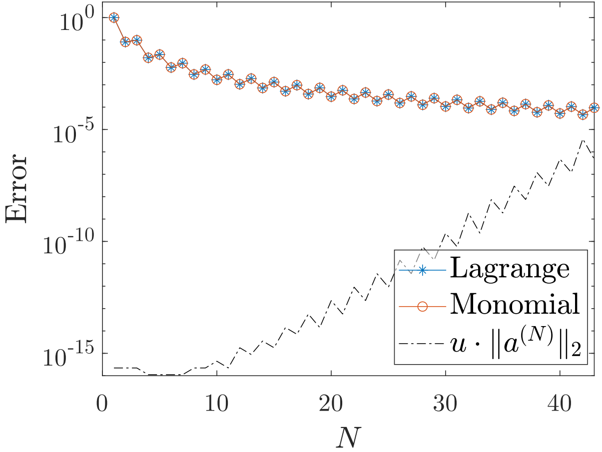

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- For example, when will the extra error (i.e., $u \cdot \|a^{(N)}\|_2$) be small?
- This requires an a priori estimate for the growth of $\|a^{(N)}\|_2$.

An important constant

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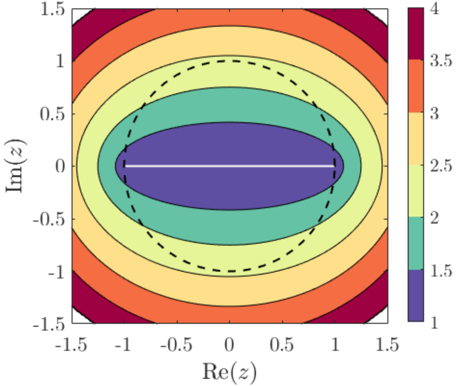

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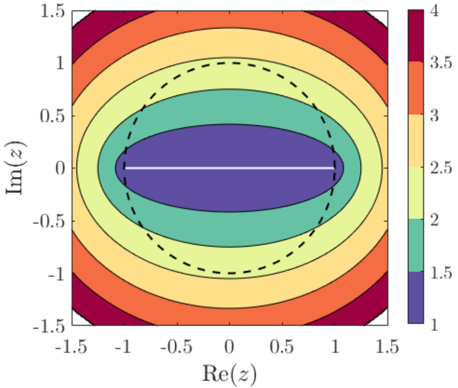

An improved upper bound for $\|a^{(N)}\|_2$

Theorem

Suppose that there exists a finite sequence of polynomials $\{Q_n\}_{n=0,1,\ldots,N}$, where Q_n has degree n, which satisfies

$$\|F - Q_n\|_{L^\infty(\Gamma)} \leq C\rho_*^{-n}, \quad 0 \leq n \leq N,$$

for some constant $C \geq 0$. The 2-norm of the monomial coefficient vector of the Nth degree interpolating polynomial P_N of F satisfies

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$$\|a^{(N)}\|_2 \lesssim C \cdot N \approx N.$$

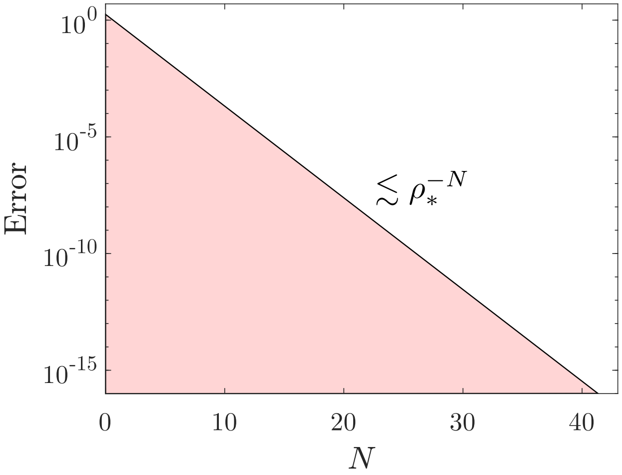
Implications: when $\|F - P_N\|_{L^\infty(\Gamma)}$ decays quickly

Therefore, when $\|(V^{(N)})^{-1}\|_2 \lesssim \frac{1}{u}$, the monomial approximation error satisfies

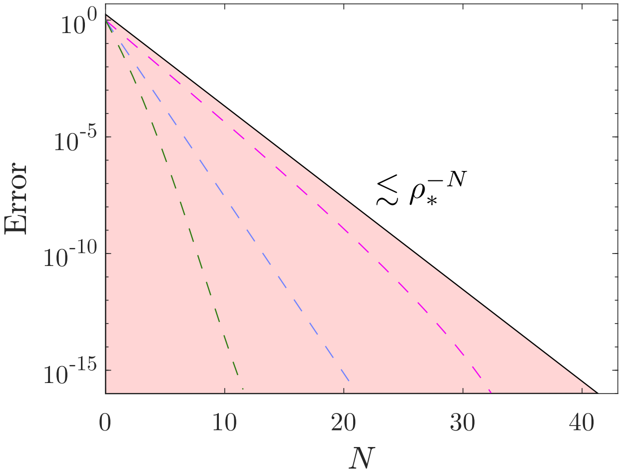
$$\|F - \widehat{P}_N\|_{L^\infty(\Gamma)} \lesssim \|F - P_N\|_{L^\infty(\Gamma)} + u \cdot N.$$

The extra error is around machine epsilon in this case!

Visualization: when $\|F - P_N\|_{L^\infty(\Gamma)}$ decays quickly

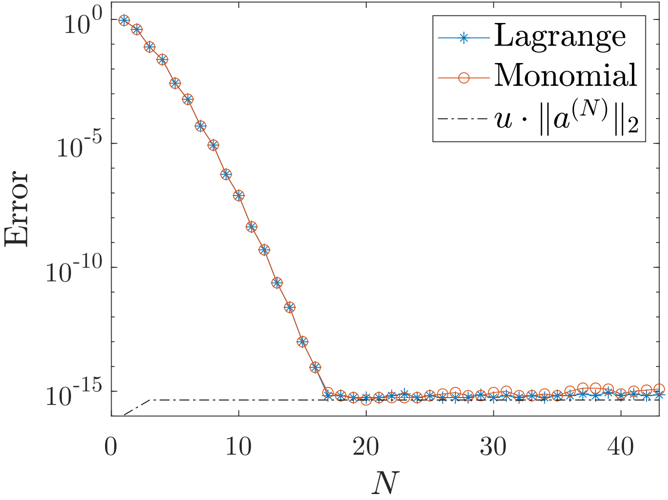


Visualization: when $\|F - P_N\|_{L^\infty(\Gamma)}$ decays quickly



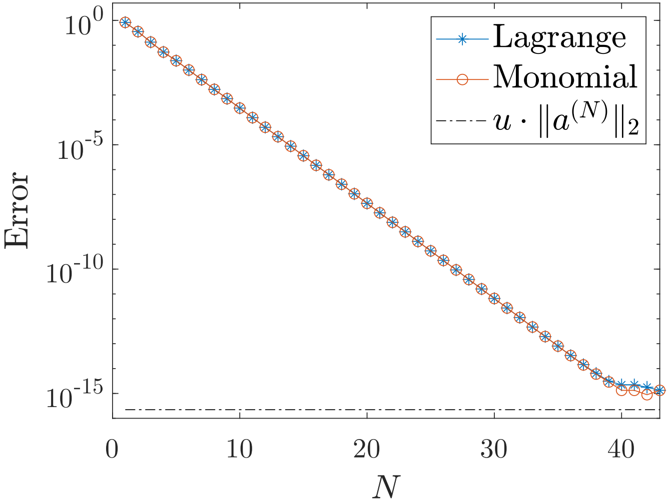
Examples: when $\|F - P_N\|_{L^\infty(\Gamma)}$ decays quickly

$$F(x) = \cos(2x + 1)$$



Examples: when $\|F - P_N\|_{L^\infty(\Gamma)}$ decays quickly

$$F(x) = \frac{1}{x-\sqrt{2}}$$



Implications: when $\|F - P_N\|_{L^\infty(\Gamma)}$ decays slowly

When $\|F - P_n\|_{L^\infty(\Gamma)} \lesssim \rho_*^{-n}$ for $0 \le n \le N$,

- the growth of $\|a^{(N)}\|_2$ is suppressed,
- and one loses nothing by using the monomial basis.

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- $\|a^{(N)}\|_2$ will be larger.
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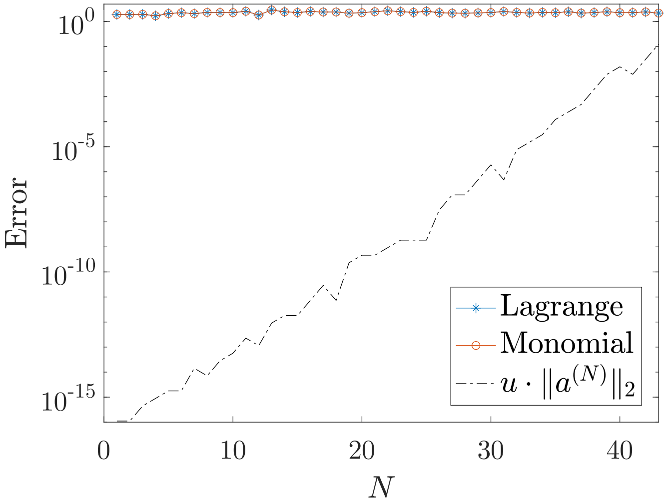
Does it matter?

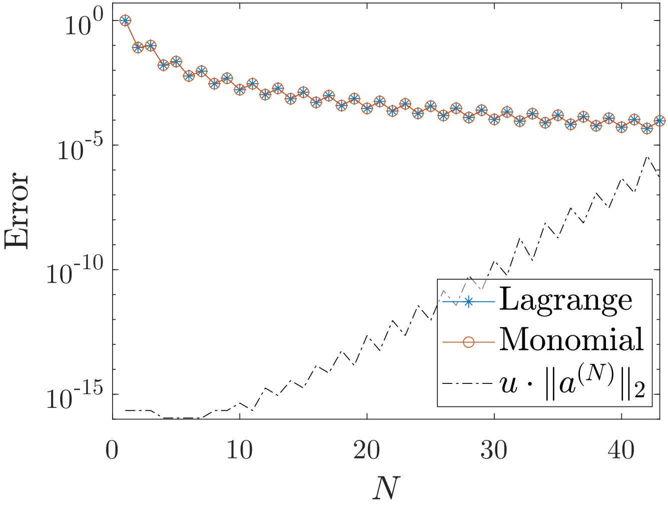
Examples: when $\|F - P_N\|_{L^\infty(\Gamma)}$ decays slowly

$$F(x) = \cos(120x + 1)$$



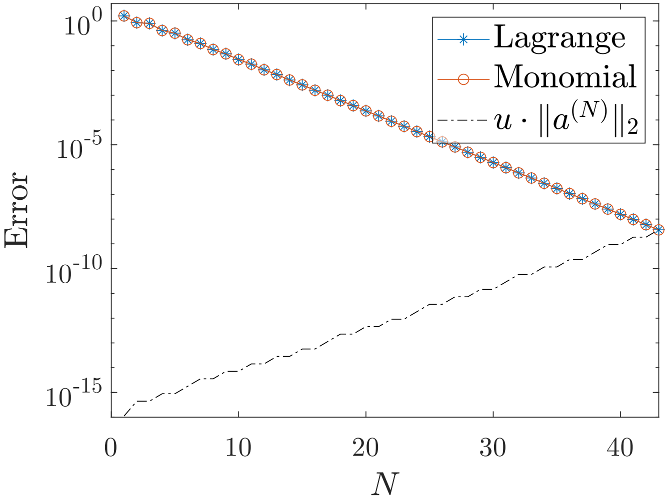
Examples: when $\|F - P_N\|_{L^\infty(\Gamma)}$ decays slowly

$$F(x) = |x|^{5/2}$$



Examples: when $\|F - P_N\|_{L^\infty(\Gamma)}$ decays slowly

$$F(x) = \frac{1}{x - 0.5i}$$



Implications: when $\|F - P_N\|_{L^\infty(\Gamma)}$ decays slowly

I'll now characterize what we just observed.

Assume that $\|F - P_n\|_{L^\infty(\Gamma)}$ decays to the value $\|F - P_N\|_{L^\infty(\Gamma)}$ at a rate slower than ρ_*^{-n}, i.e.,

$$\|F - P_n\|_{L^\infty(\Gamma)} \leq \rho_*^{N-n} \|F - P_N\|_{L^\infty(\Gamma)}, \quad \text{for } 0 \leq n \leq N.$$

Visualizations: when $\|F - P_N\|_{L^\infty(\Gamma)}$ decays slowly

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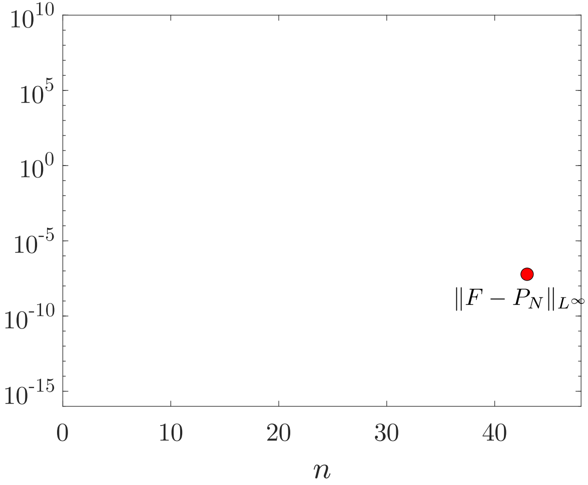

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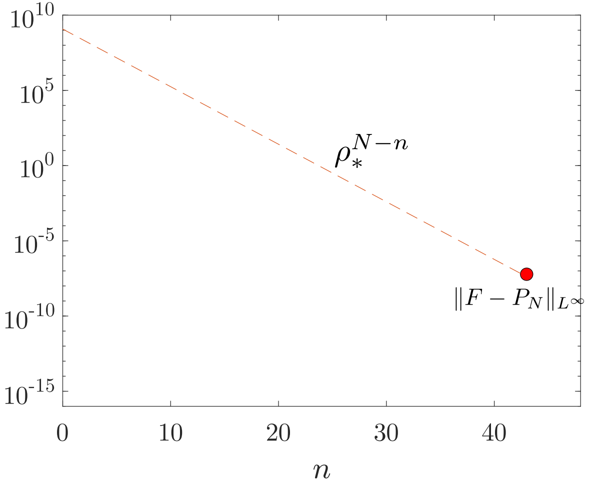

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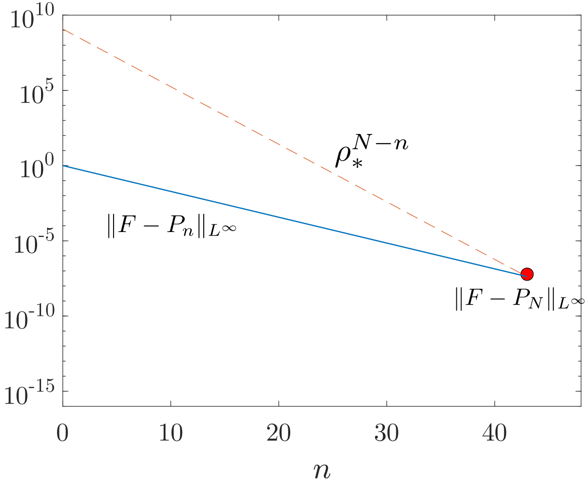

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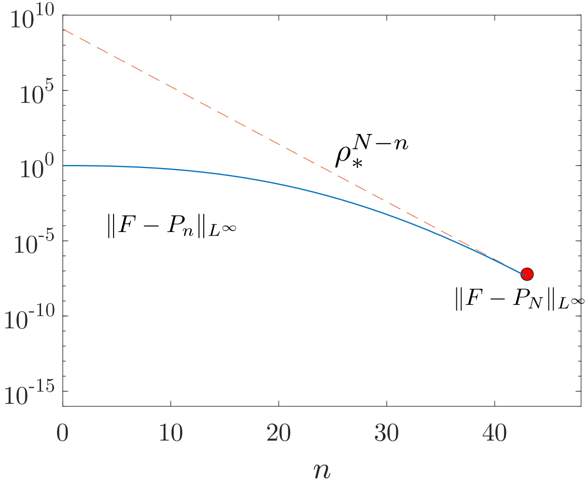

Implications: when $\|F - P_N\|_{L^\infty(\Gamma)}$ decays slowly

Theorem

Under this assumption, the monomial approximation error satisfies

$$\|F - \widehat{P}_N\|_{L^\infty(\Gamma)} \lesssim 2\|F - P_N\|_{L^\infty(\Gamma)},$$

so long as $\|(V^{(N)})^{-1}\|_2 \lesssim \frac{1}{u}$.

The proof is similar to the previous case.

Implications: stagnation of convergence

We've shown that if $\|F - P_n\|_{L^\infty(\Gamma)}$

- decays at a rate **faster** than ρ_*^{-n},
- or decays at a rate **slower** than ρ_*^{-n},

then the monomial basis = a well-conditioned basis when the order $\leq$ threshold.

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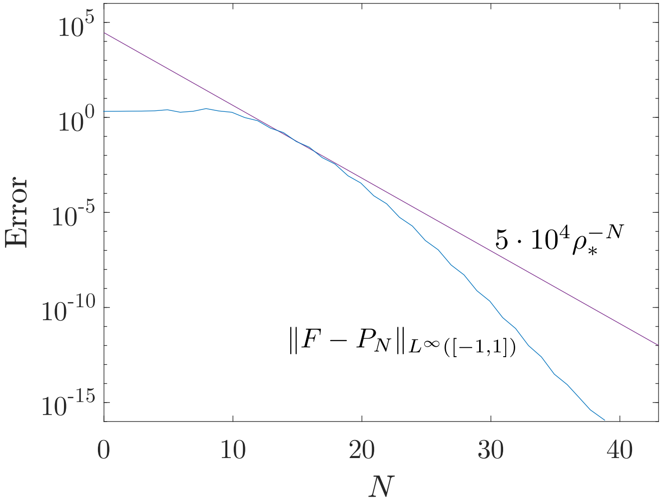
The only way for stagnation to happen before the order reaches the threshold is that, $\|F - P_n\|_{L^\infty(\Gamma)}$ first decays at a rate **slower** than ρ_*^{-n}, then starts to decay at a rate **faster** than ρ_*^{-n}.

Examples: stagnation of convergence

$$F(x) = \cos(12x + 1)$$



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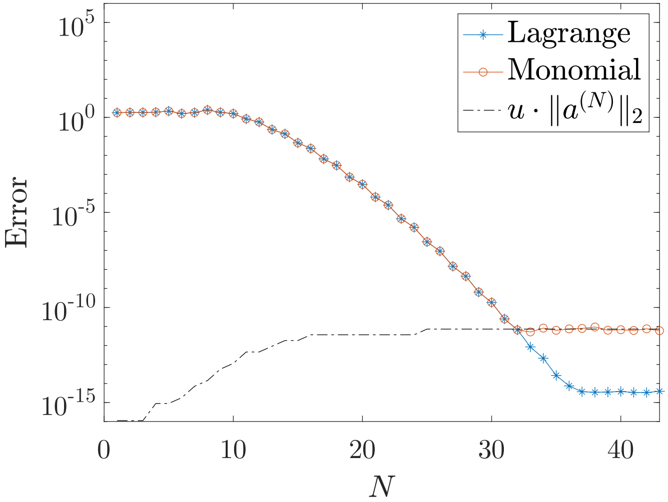
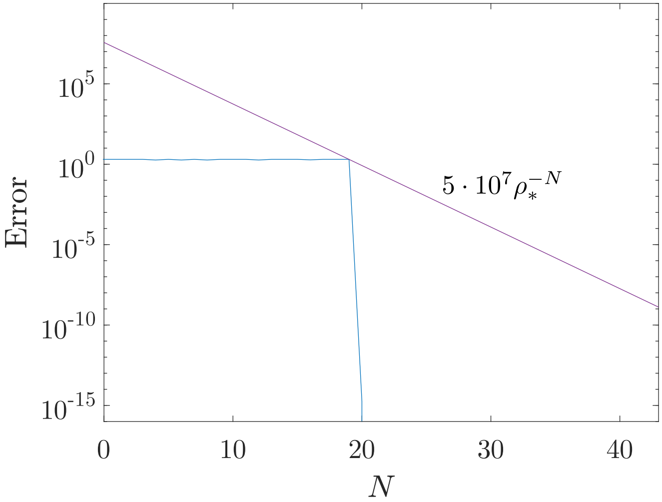


Examples

$$F(x) = T_{20}(x)$$



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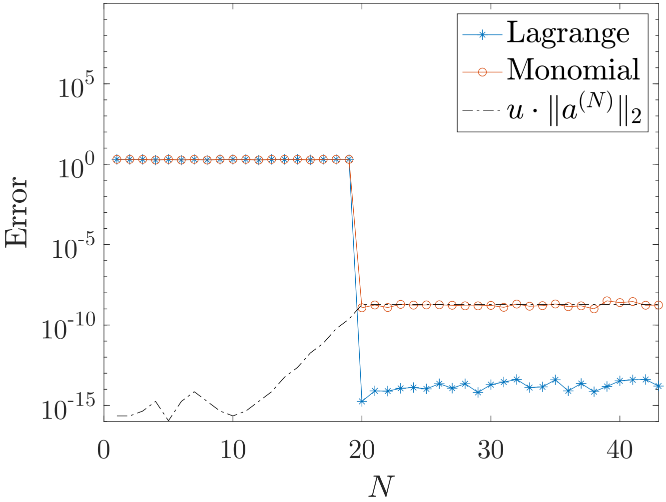

How restrictive is the monomial basis?

- Extremely high-order interpolation is impossible due to the precondition $\|(V^{(N)})^{-1}\|_2 \lesssim \frac{1}{u}$.
- So **global** interpolation won't work.

How restrictive is the monomial basis?

On the other hand, **piecewise** polynomial interpolation in the monomial basis over a partition of Γ can be carried out stably, provided that

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The convergence rate of piecewise polynomial approximation is $\mathcal{O}(h^{N+1})$.

Conclusions

There are many other applications of this work (see our paper).

This paper is not only about monomials. It characterizes the universal behavior of function approximation with any ill-conditioned basis before the condition number reaches $1/u$.

Paper & slides are available on my personal website (`https://zewenshen.github.io`).

Thank you for listening!

Bonus

How restrictive is the monomial basis?

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- Highly optimized linear algebra libraries, e.g., LAPACK.
- $\mathcal{O}(N^2)$ algorithms exist (could be less backward stable).

Generalization to higher dimensions

In 2-D, the Vandermonde matrix looks like

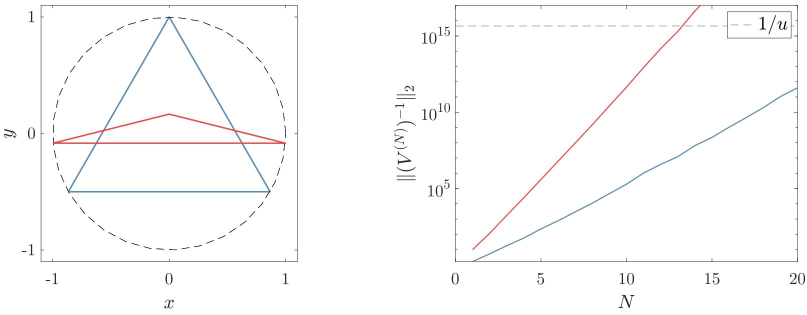
$$V^{(N)} := \begin{pmatrix} 1 & x_1 & y_1 & x_1^2 & x_1 y_1 & \cdots & y_1^N \\ 1 & x_2 & y_2 & x_2^2 & x_2 y_2 & \cdots & y_2^N \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{\widetilde{N}} & y_{\widetilde{N}} & x_{\widetilde{N}}^2 & x_{\widetilde{N}} y_{\widetilde{N}} & \cdots & y_{\widetilde{N}}^N \end{pmatrix},$$

where $\widetilde{N}$ is the dimensionality of bivariate polynomials of order up to N.

Collocation points with relatively small Lebesgue constants have been constructed (Vioreanu & Rokhlin 2014).

The theory of monomial approximation is essentially same as 1-D.

Numerical experiments



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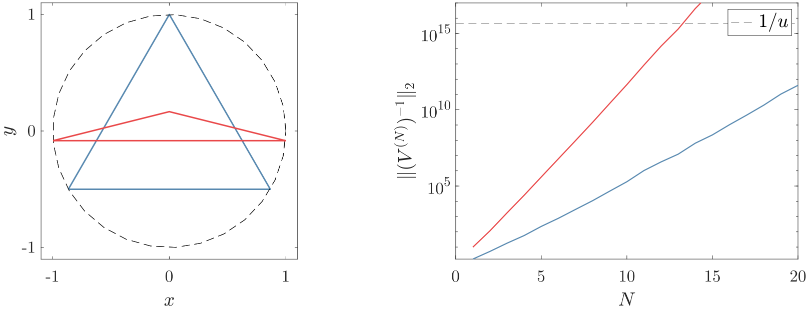
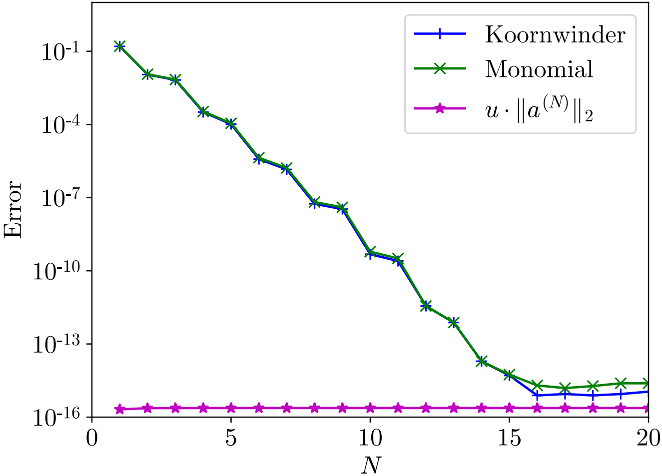


I'll show some experiments that compares the monomial basis with the Koornwinder polynomial basis over the blue triangle.

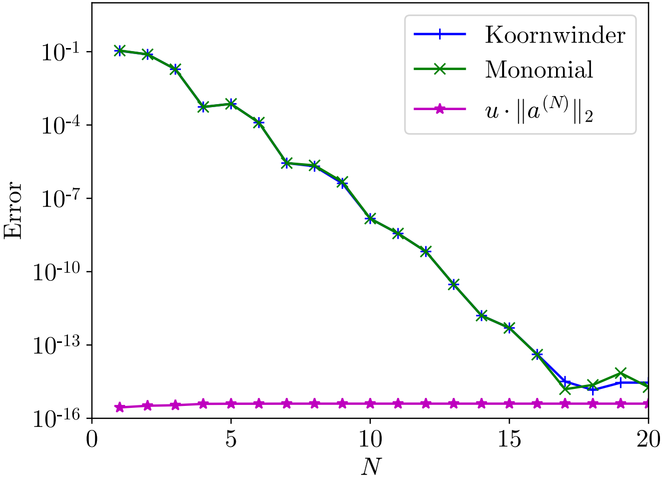
Numerical experiments

$$F(x, y) = e^{-(x^2+y^2)/4}$$



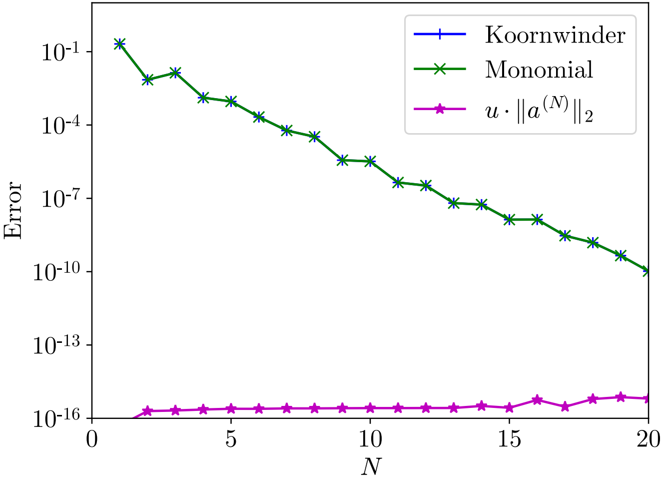
Numerical experiments

$$F(x, y) = \sin(xy/2 + x + y)$$



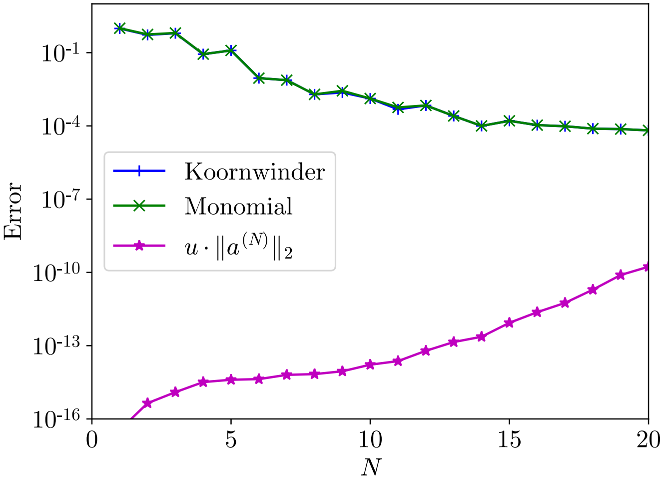
Numerical experiments

$$F(x, y) = \arctan(x) \cdot \arctan(y)$$



Numerical experiments

$$F(x, y) = |x + y|^{5.5}$$



Bonus: what happens when the order > the threshold?

$\cos(12x + 1)$, MATLAB's backslash

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