Is polynomial interpolation in the monomial basis unstable?

Zewen Shen, Kirill Serkh

University of Toronto

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Oscillatory integrals

Let $\omega \in \mathbb{R}$, and let $F : [-1, 1] \to \mathbb{R}$ be a smooth function.

$$\int_{-1}^{1} e^{i\omega x} F(x) \, \mathrm{d}x$$

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$$\int_{-1}^{1} e^{i\omega x} x^{k+1} \, \mathrm{d}x = \frac{1}{i\omega}\Big(e^{i\omega} + (-1)^k e^{-i\omega} - (k+1) \int_{-1}^{1} e^{i\omega x} x^k \, \mathrm{d}x\Big)$$

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If the function F is approximated by a monomial expansion, the computational cost is independent of ω.

Singular integrals (Helsing & Ojala 2008)

Let $\Gamma \subset \mathbb{C}$ be a smooth curve. Given an analytic function $F : \Gamma \to \mathbb{C}$ and a point $\xi \in \mathbb{C}$ close to Γ,

$$\int_\Gamma \frac{F(z)}{z - \xi} \, \mathrm{d}z$$

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$$\int_\Gamma \frac{1}{z - \xi} \, \mathrm{d}z = \log(z_1 - \xi) - \log(z_0 - \xi) + 2\pi i \mathcal{N}_\xi$$

$$\int_\Gamma \frac{z^k}{z - \xi} \, \mathrm{d}z = \frac{1 - (-1)^{k-1}}{k - 1} + \xi \int_\Gamma \frac{z^{k-1}}{z - x} \, \mathrm{d}z$$

Hadamard finite-part integral

Given $\nu \in \mathbb{R}$ and $M \in \mathbb{N}_{\geq 0}$, we're interested in the calculation of the Hadamard finite-part integral

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When $F = x^m y^n$ for some $m \geq 2$, $n \geq 2$,

$$\begin{aligned} \nabla^{-2}[x^m y^n] &= \frac{x^{m+2} y^n}{(m+2)(m+1)} - \frac{n(n-1)}{(m+2)(m+1)} \nabla^{-2}[x^{m+2} y^{n-2}] \\ &= \frac{x^m y^{n+2}}{(n+2)(n+1)} - \frac{m(m-1)}{(n+2)(n+1)} \nabla^{-2}[x^{m-2} y^{n+2}] \end{aligned}$$

Motivations

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- General attitude has remained skeptical.

Polynomial interpolation

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Given a function $F : [-1, 1] \to \mathbb{C}$, the Nth degree interpolating polynomial P_N of F satisfies $P_N(x_j) = F(x_j)$, for a set of $(N+1)$ distinct collocation points $\{x_j\}_{j=0,1,\ldots,N}$.

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The choice of collocation points is important for good approximation quality.

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- Polynomial interpolation can be viewed as a linear operator from $\big(F(x_0), F(x_1), \ldots, F(x_N)\big)$ to the interpolating polynomial P_N.

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We'll choose collocation points with small Λ_N.

Polynomial interpolation in finite precision

To compute P_N on a computer, we first choose a polynomial basis $\{\phi_k\}_k$

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- Condition number
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The standard choices:

- Lagrange polynomials.
- Orthogonal polynomials (Chebyshev, Legendre, etc).

Polynomial interpolation in the monomial basis

What about expressing P_N in the monomial basis?

$$P_N(x) = \sum_{k=0}^{N} a_k x^k$$

The previous linear system becomes

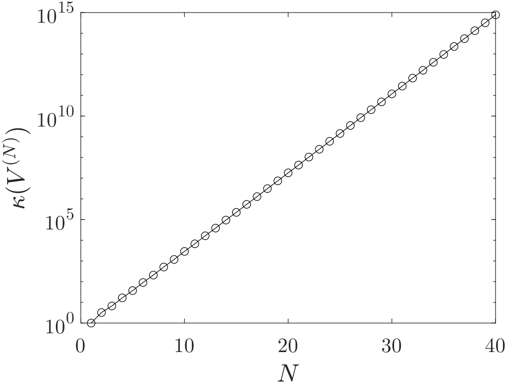
$$\underbrace{\begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^N \\ 1 & x_1 & x_1^2 & \cdots & x_1^N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_N & x_N^2 & \cdots & x_N^N \end{pmatrix}}_{V^{(N)}} \underbrace{\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_N \end{pmatrix}}_{a^{(N)}} = \underbrace{\begin{pmatrix} F(x_0) \\ F(x_1) \\ \vdots \\ F(x_N) \end{pmatrix}}_{f^{(N)}}.$$

$V^{(N)}$ is known as a Vandermonde matrix.

Monomial basis is ill-conditioned

Given any set of real collocation points, $\kappa(V^{(N)})$ grows at least exponentially fast.

Example: when the Chebyshev points are used for collocation:



Numerical experiments

Let's run some experiments. The following quantities will be reported.

- $\|F - \widehat{P}_N\|_{L^\infty([-1,1])}$: Monomial approximation error.
  Denoted by the label "monomial".
- $\|F - P_N\|_{L^\infty([-1,1])}$: Exact polynomial interpolation error, estimated using the Barycentric interpolation formula.
  Denoted by the label "Lagrange".

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Numerical experiments

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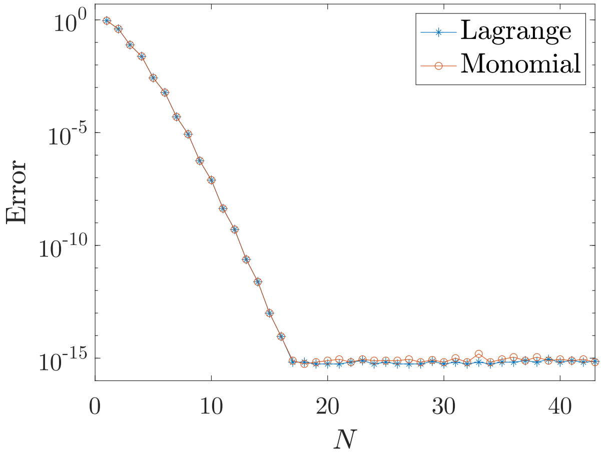

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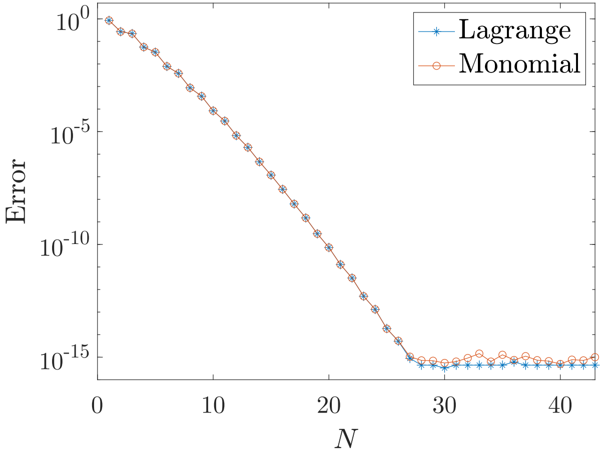
$$F(x) = e^{-2(x+0.1)^2}$$



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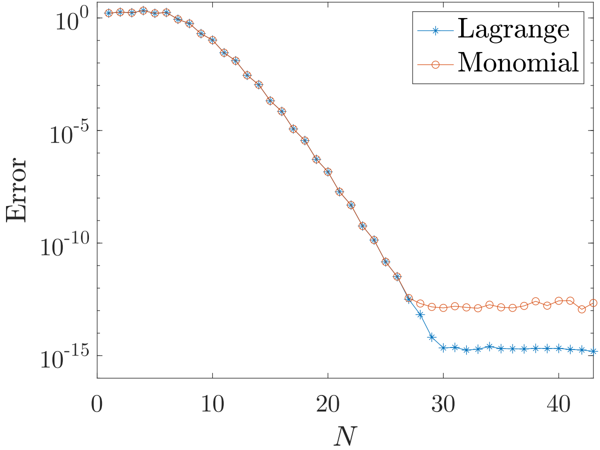
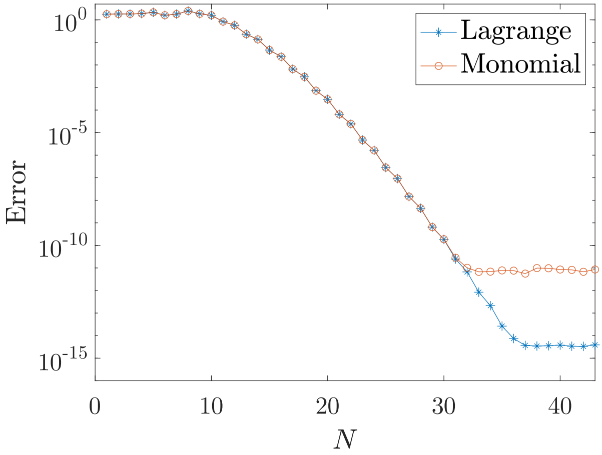


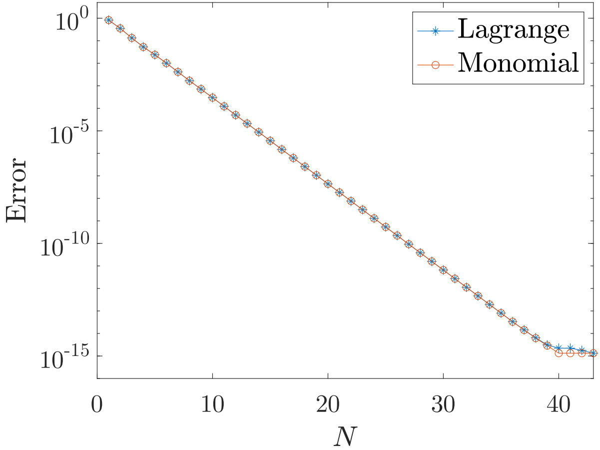
Numerical experiments

$$F(x) = \cos(12x + 1)$$



Numerical experiments

$$F(x) = \frac{1}{x-\sqrt{2}}$$



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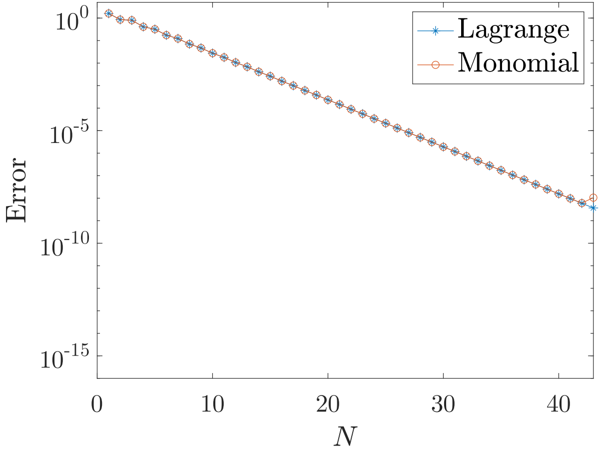
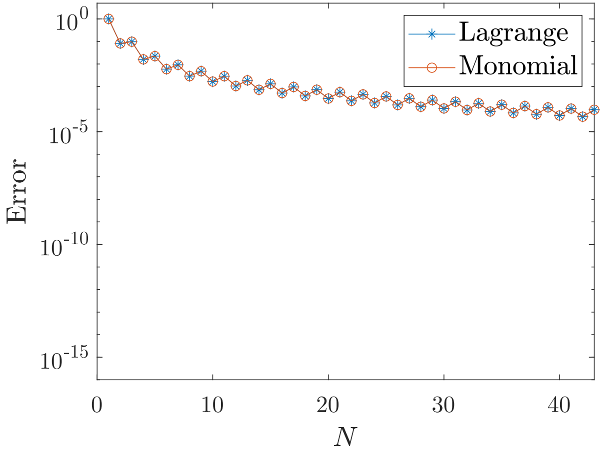
$$F(x) = \frac{1}{x-0.5i}$$



Numerical experiments

$$F(x) = |x|^{2.5}$$



Reflections

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Before I explain why, I'll present my favorite application.

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On the other hand, this is what a monomial look like:

$$x^m y^n.$$

Works for any domain. Much more handy. Much cheaper to evaluate.

Assumptions

In the rest of the talk, here's a list of our assumptions:

- The domain of approximation $\Gamma \subset \mathbb{C}$ can be an arbitrary smooth simple arc.
- Γ is inside the unit disk D_1 centered at the origin.
- The Lebesgue constant Λ_N of the collocation points is small.

Feel free to consider $\Gamma = [-1, 1]$ with Chebyshev points.

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$$\Longrightarrow$$

extremely inaccurate monomial coefficients

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What's really important is the backward error, i.e.,

$$\|V^{(N)}\widehat{a}^{(N)} - f^{(N)}\|_2,$$

of the numerical solution $\widehat{a}^{(N)}$ to the Vandermonde system $V^{(N)}a^{(N)} = f^{(N)}$.

Rethinking interpolation

The difference between the exact interpolating polynomial P_N and the computed monomial expansion $\widehat{P}_N$ satisfies

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How large will the backward error be?

Backward stable linear system solver

When a backward stable linear system solver is used to solve the Vandermonde system $V^{(N)}a^{(N)} = f^{(N)}$, the numerical solution $\hat{a}^{(N)}$ is the exact solution to

$$\left(V^{(N)} + \delta V^{(N)}\right)\hat{a}^{(N)} = f^{(N)},$$

for some $\delta V^{(N)} \in \mathbb{C}^{(N+1)\times(N+1)}$ that satisfies

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where u denotes machine epsilon and $\gamma_N = \mathcal{O}(\|V^{(N)}\|_2)$.

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It follows that

$$\|V^{(N)}\widehat{a}^{(N)} - f^{(N)}\|_2 = \|\delta V^{(N)}\widehat{a}^{(N)}\|_2 \leq u \cdot \gamma_N \|\widehat{a}^{(N)}\|_2.$$

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Recall that $\|\delta V^{(N)}\|_2 \leq u \cdot \gamma_N$. If $\|(V^{(N)})^{-1}\|_2 \leq \frac{1}{2u \cdot \gamma_N}$, we have that

$$\big\|\big(V^{(N)}\big)^{-1}\big(\delta V^{(N)}\big)\big\|_2 \leq \big\|\big(V^{(N)}\big)^{-1}\big\|_2 \cdot \big\|\delta V^{(N)}\big\|_2 \leq \frac{1}{2}.$$

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It follows from the geometric series theorem that $\|\widehat{a}^{(N)}\|_2 \leq 2\|a^{(N)}\|_2$.

Monomial approximation error

We've shown that

$$\|P_N - \widehat{P}_N\|_{L^\infty(\Gamma)} \leq \Lambda_N \|V^{(N)}\widehat{a}^{(N)} - f^{(N)}\|_2,$$

and that

$$\|(V^{(N)})^{-1}\|_2 \leq \frac{1}{2u \cdot \gamma_N} \implies \left\|V^{(N)}\widehat{a}^{(N)} - f^{(N)}\right\|_2 \leq 2u \cdot \gamma_N \|a^{(N)}\|_2.$$

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$$\|P_N - \widehat{P}_N\|_{L^\infty(\Gamma)} \leq \Lambda_N \|V^{(N)}\widehat{a}^{(N)} - f^{(N)}\|_2,$$

and that

$$\|(V^{(N)})^{-1}\|_2 \leq \frac{1}{2u \cdot \gamma_N} \implies \left\|V^{(N)}\widehat{a}^{(N)} - f^{(N)}\right\|_2 \leq 2u \cdot \gamma_N \|a^{(N)}\|_2.$$

Assume that $\|(V^{(N)})^{-1}\|_2 \leq \frac{1}{2u \cdot \gamma_N}$. By the triangle inequality, the monomial approximation error satisfies

$$\|F - \widehat{P}_N\|_{L^\infty(\Gamma)} \leq \|F - P_N\|_{L^\infty(\Gamma)} + \|P_N - \widehat{P}_N\|_{L^\infty(\Gamma)}$$

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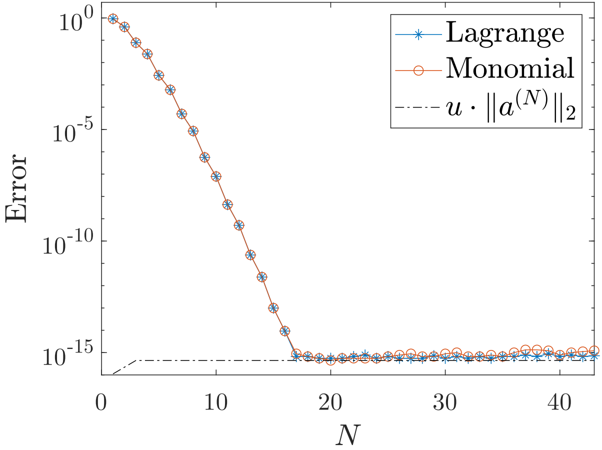
- Extra additive error term $\approx u \cdot \|a^{(N)}\|_2$.

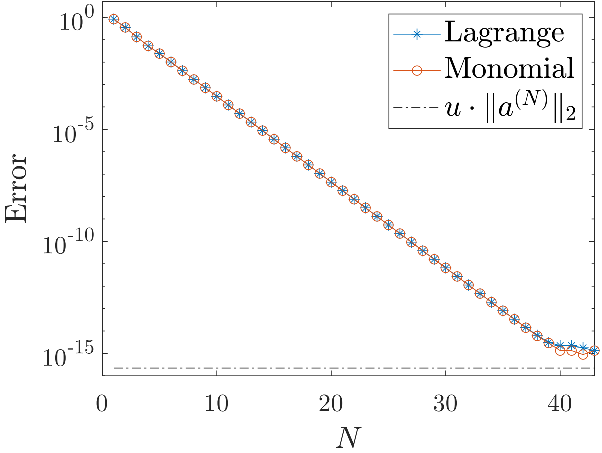
Numerical experiments

$$F(x) = \cos(2x + 1)$$



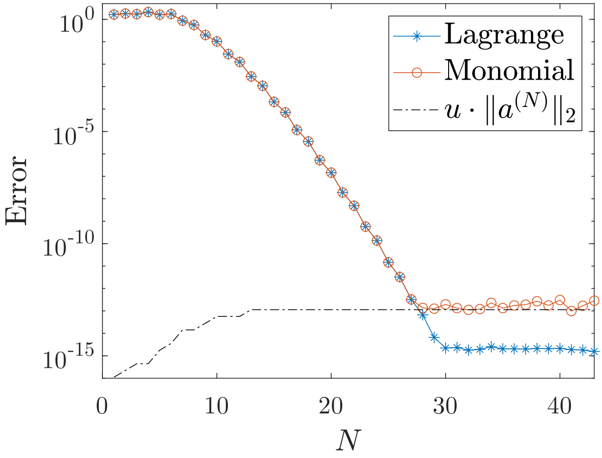
Numerical experiments

$$F(x) = \frac{1}{x - \sqrt{2}}$$



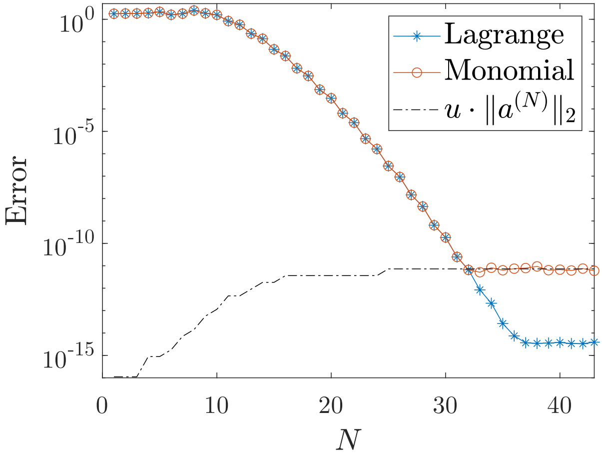
Numerical experiments

$$F(x) = \cos(8x + 1)$$



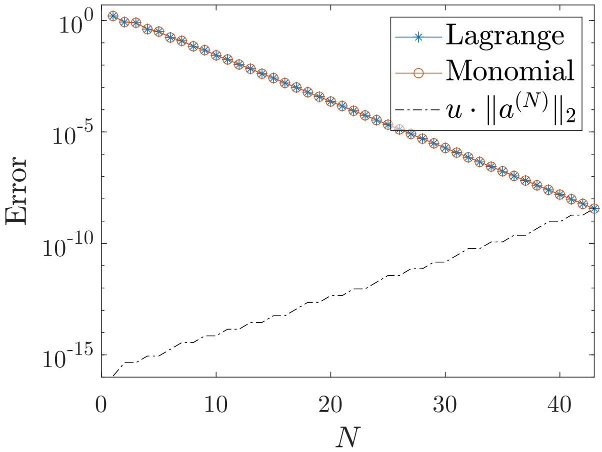
Numerical experiments

$$F(x) = \cos(12x + 1)$$



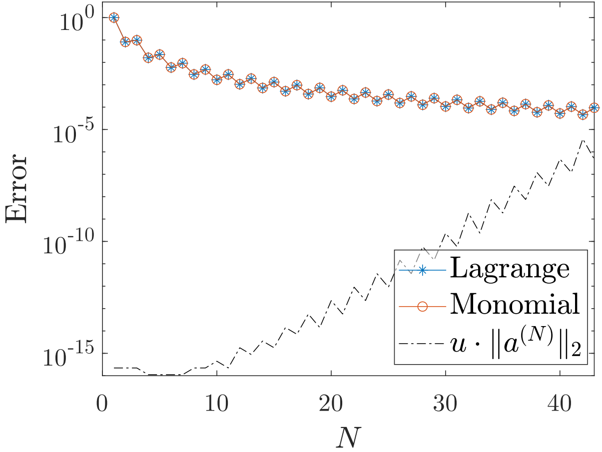
Numerical experiments

$$F(x) = \frac{1}{x - 0.5i}$$



Numerical experiments

$$F(x) = |x|^{2.5}$$



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Remark: The literature sometimes cites backward stability as the only justification for the use of the monomial basis, which is somewhat misguided.

Monomial coefficients of P_N

Lemma

Let $P_N : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree N, where $P_N(z) = \sum_{k=0}^{N} a_k z^k$ for some $a_0, a_1, \ldots, a_N \in \mathbb{C}$. The 2-norm of the coefficient vector $a^{(N)} := (a_0, a_1, \ldots, a_N)^T$ satisfies

$$\|a^{(N)}\|_2 \leq \|P_N\|_{L^\infty(\partial D_1)},$$

where D_1 denotes the open unit disk centered at the origin.

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Proof.

Observe that $P_N(e^{i\theta}) = \sum_{k=0}^{N} a_k e^{ik\theta}$. Thus, by Parseval's identity, we have that $\|a^{(N)}\|_2 = \left(\frac{1}{2\pi} \int_0^{2\pi} |P_N(e^{i\theta})|^2 \, \mathrm{d}\theta\right)^{1/2} \leq \|P_N\|_{L^\infty(\partial D_1)}$. □

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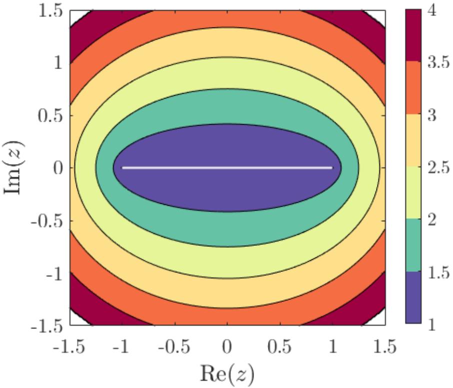
- Very tight bound, but relies on knowledge of $\|P_N\|_{L^\infty(\partial D_1)}$.

Bernstein ellipse

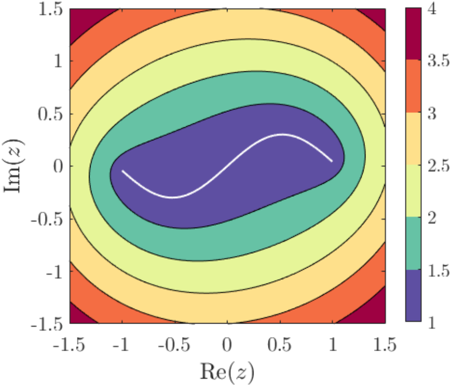
Given $\rho > 1$, the Bernstein ellipse E_ρ for $\Gamma = [-1, 1]$ is the image of a circle centered at the origin with radius ρ under the mapping $\frac{1}{2}(z + \frac{1}{z})$.



- The larger ρ, the bigger the ellipse.
- We let E_ρ^o denote the open region bounded by E_ρ.

Generalization of Bernstein ellipse

The concept of the Bernstein ellipse can be generalized to an arbitrary smooth simple arc $\Gamma \subset \mathbb{C}$.



Bernstein's inequality

Lemma (Walsh 1935)

Let Γ be a smooth simple arc in the complex plane, and let E_ρ^o be the region corresponding to Γ with some parameter $\rho > 1$. Then, the L^∞ norm of any polynomial P_N of degree N over E_ρ^o satisfies

$$\|P_N\|_{L^\infty(E_\rho^o)} \leq \rho^N \|P_N\|_{L^\infty(\Gamma)}.$$

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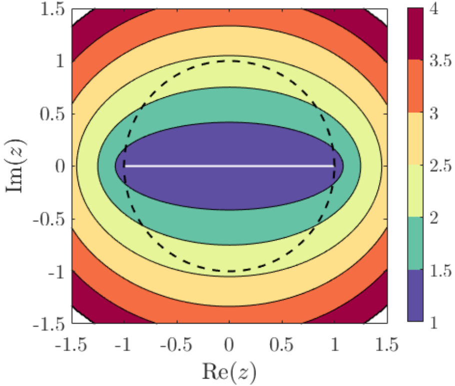
Define $\rho_* = \min\{\rho > 1 : D_1 \subset E_\rho^o\}$. Then,

$$\|a^{(N)}\|_2 \leq \|P_N\|_{L^\infty(\partial D_1)} \leq \|P_N\|_{L^\infty(E_{\rho_*}^o)} \leq \rho_*^N \|P_N\|_{L^\infty(\Gamma)}.$$

Examples of ρ_*

Example

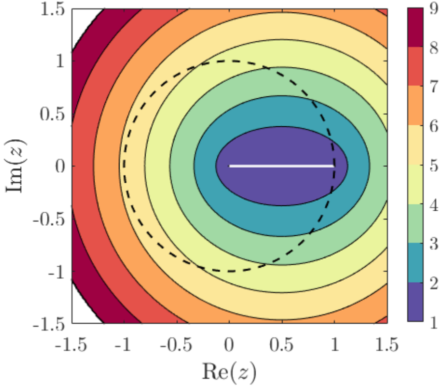
When $\Gamma = [-1, 1]$, $\rho_* = 1 + \sqrt{2} \approx 2.4$



Examples of ρ_*

Example

When $\Gamma = [0,1]$, $\rho_* = 3 + 2\sqrt{2} \approx 5.8$



An upper bound for $\|(V^{(N)})^{-1}\|_2$

We've shown that

$$\frac{\|a^{(N)}\|_2}{\|P_N\|_{L^\infty(\Gamma)}} \leq \rho_*^N.$$

Also note that

$$\|(V^{(N)})^{-1}\|_2 = \sup_{f^{(N)}\neq 0}\left\{\frac{\|(V^{(N)})^{-1}f^{(N)}\|_2}{\|f^{(N)}\|_2}\right\} = \sup_{f^{(N)}\neq 0}\left\{\frac{\|a^{(N)}\|_2}{\|f^{(N)}\|_2}\right\}.$$

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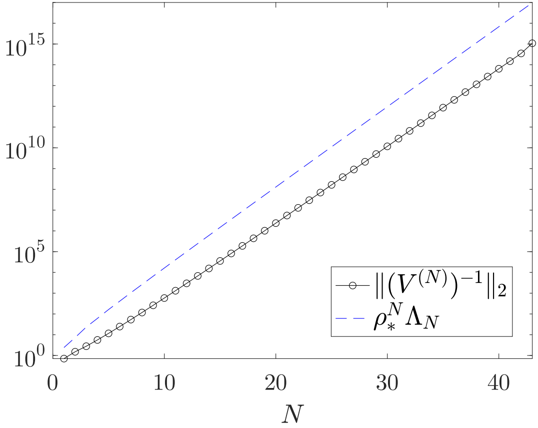
Theorem (Shen & Serkh (2022))

Suppose that $V^{(N)} \in \mathbb{C}^{(N+1)\times(N+1)}$ is a Vandermonde matrix with $(N+1)$ distinct collocation points over $\Gamma \subset \mathbb{C}$. Then,

$$\|(V^{(N)})^{-1}\|_2 \leq \rho_*^N \Lambda_N.$$

An upper bound for $\|(V^{(N)})^{-1}\|_2$

$\Gamma = [-1,1]$, $\rho_* = 1+\sqrt{2}$, Chebyshev points



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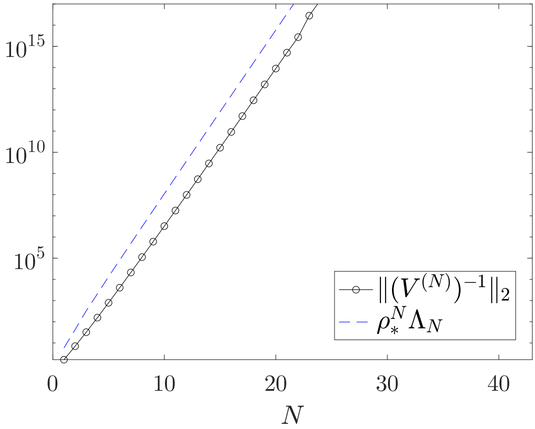

An upper bound for $\|a^{(N)}\|_2$

Theorem (Shen & Serkh (2022))

Suppose that there exists a finite sequence of polynomials $\{Q_n\}_{n=0,1,\dots,N}$, where Q_n has degree n, which satisfies

$$\|F - Q_n\|_{L^\infty(\Gamma)} \le C\rho^{-n}, \quad 0 \le n \le N,$$

for some constants $\rho > 1$ and $C \ge 0$. The 2-norm of the monomial coefficient vector of the Nth degree interpolating polynomial P_N of F satisfies

$$\|a^{(N)}\|_2 \le \|F\|_{L^\infty(\Gamma)} + C\left(\Lambda_N\left(\frac{\rho_*}{\rho}\right)^N + 2\rho_* \sum_{j=0}^{N-1}\left(\frac{\rho_*}{\rho}\right)^j + 1\right).$$

A simplified upper bound for $\|a^{(N)}\|_2$

We fix the variable ρ to be ρ_*. The previous theorem becomes:

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We first deal with the case where the $\|F - P_n\|_{L^\infty(\Gamma)} \lesssim \rho_*^{-n}$.

$$\|a^{(N)}\|_2 \lesssim C \cdot N \approx N.$$

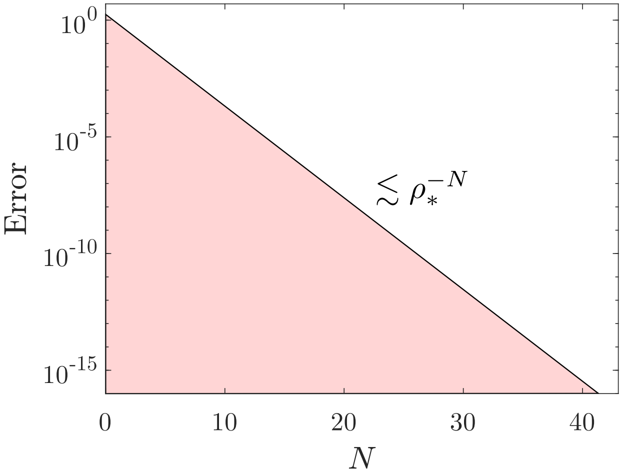
Implications: when $\|F - P_N\|_{L^\infty(\Gamma)}$ decays quickly

Therefore, when $\|(V^{(N)})^{-1}\|_2 \lesssim \frac{1}{u}$, the monomial approximation error satisfies

$$\|F - \widehat{P}_N\|_{L^\infty(\Gamma)} \lesssim \|F - P_N\|_{L^\infty(\Gamma)} + u \cdot N.$$

The extra error is around machine epsilon in this case!

Visualization: when $\|F - P_N\|_{L^\infty(\Gamma)}$ decays quickly



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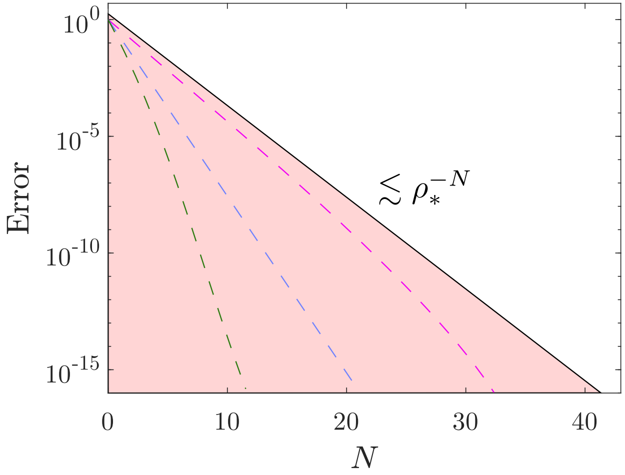

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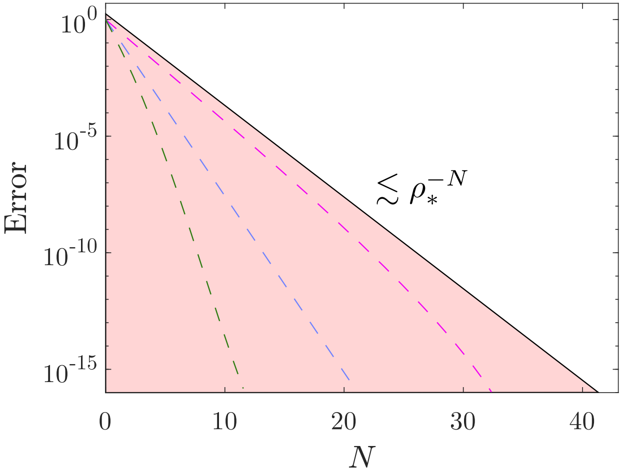

Does the precondition $\|(V^{(N)})^{-1}\|_2 \lesssim \frac{1}{u}$ weaken our result?

Implications: when $\|F - P_N\|_{L^\infty(\Gamma)}$ decays quickly

Recall that $\|(V^{(N)})^{-1}\|_2 \leq \rho_*^N \Lambda_N$.

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Recall that $\|(V^{(N)})^{-1}\|_2 \leq \rho_*^N \Lambda_N$.

When N satisfies $\rho_*^{-N} = u$, $\|(V^{(N)})^{-1}\|_2 \leq \frac{\Lambda_N}{u} \lesssim \frac{1}{u}$.

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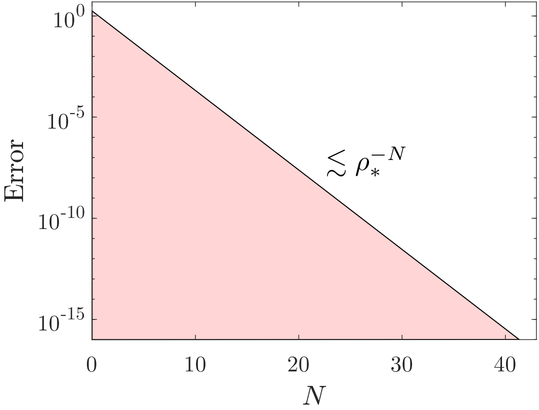
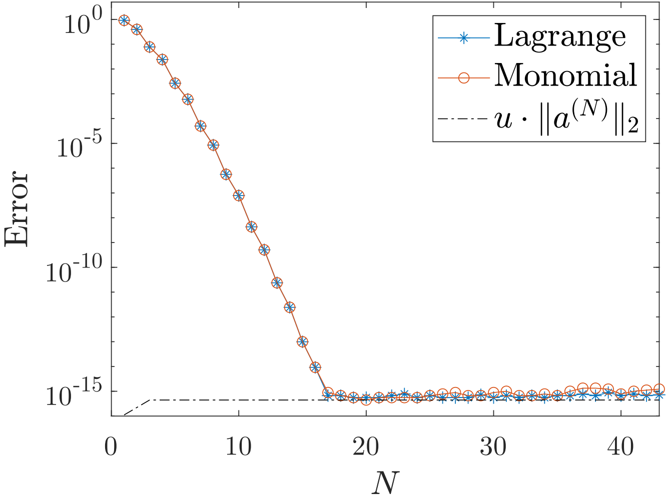


The threshold will always be on the right of this pink region.

Examples: when $\|F - P_N\|_{L^\infty(\Gamma)}$ decays quickly

$$F(x) = \cos(2x + 1)$$



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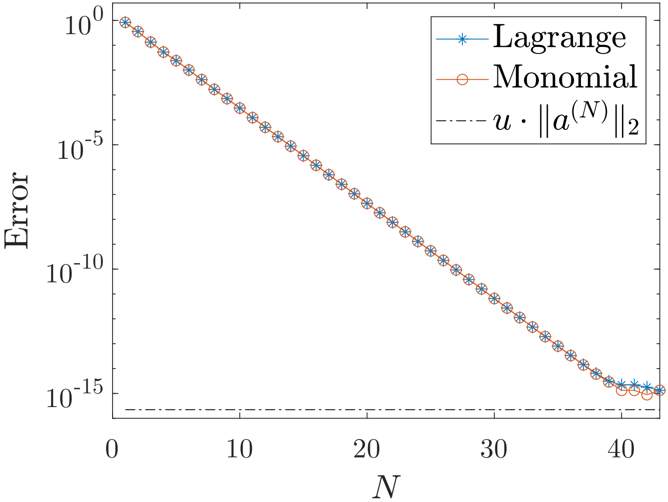

Implications: when $\|F - P_N\|_{L^\infty(\Gamma)}$ decays slowly

When $\|F - P_n\|_{L^\infty(\Gamma)} \lesssim \rho_*^{-n}$ for $0 \leq n \leq N$,

- the growth of $\|a^{(N)}\|_2$ is suppressed,
- and one loses nothing by using the monomial basis.

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What happens if the polynomial interpolation error decays more slowly?

- $\|a^{(N)}\|_2$ will be larger.
- extra error caused by the monomial basis becomes non-negligible.

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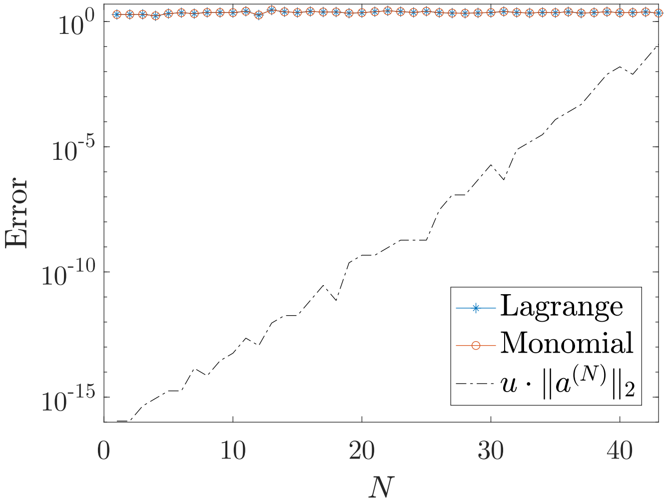
Does it matter?

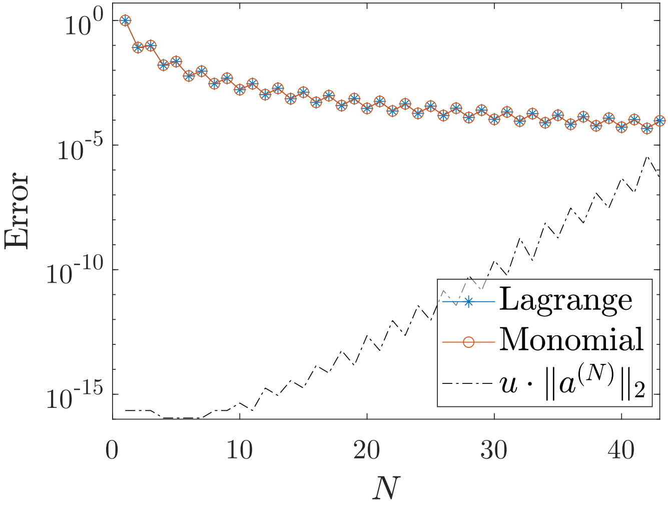
Examples: when $\|F - P_N\|_{L^\infty(\Gamma)}$ decays slowly

$$F(x) = \cos(120x + 1)$$



Examples: when $\|F - P_N\|_{L^\infty(\Gamma)}$ decays slowly

$$F(x) = |x|^{5/2}$$



Examples: when $\|F - P_N\|_{L^\infty(\Gamma)}$ decays slowly

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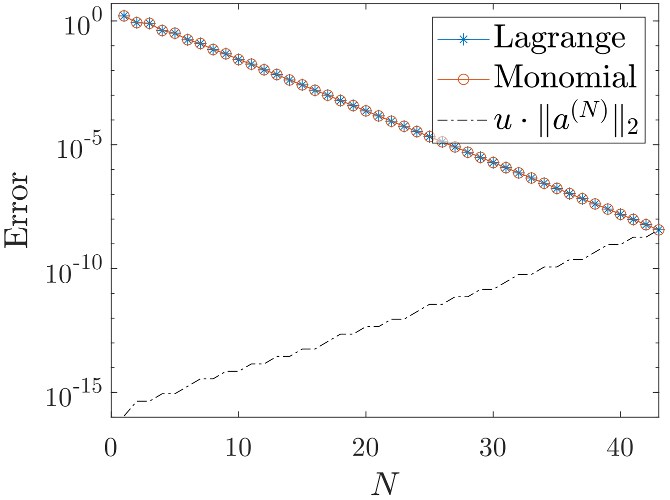

Implications: when $\|F - P_N\|_{L^\infty(\Gamma)}$ decays slowly

I'll now characterize what we just observed.

Assume that $\|F - P_n\|_{L^\infty(\Gamma)}$ decays to the value $\|F - P_N\|_{L^\infty(\Gamma)}$ at a rate slower than ρ_*^{-n}, i.e.,

$$\|F - P_n\|_{L^\infty(\Gamma)} \leq \rho_*^{N-n} \|F - P_N\|_{L^\infty(\Gamma)}, \quad \text{for } 0 \leq n \leq N.$$

Visualizations: when $\|F - P_N\|_{L^\infty(\Gamma)}$ decays slowly

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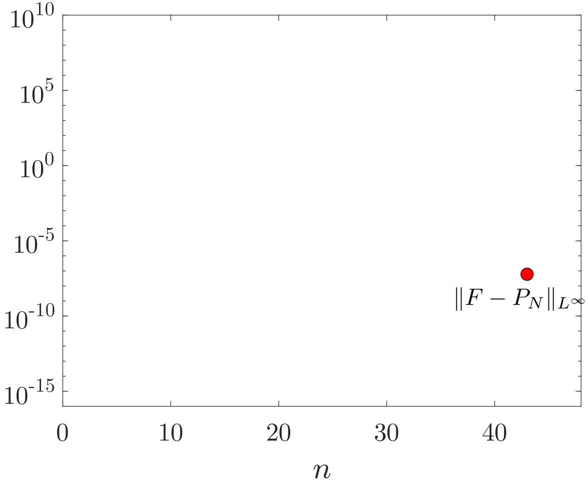

Visualizations: when $\|F - P_N\|_{L^\infty(\Gamma)}$ decays slowly

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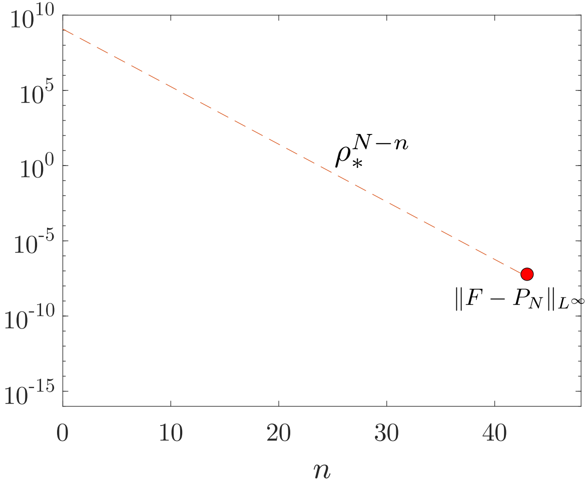

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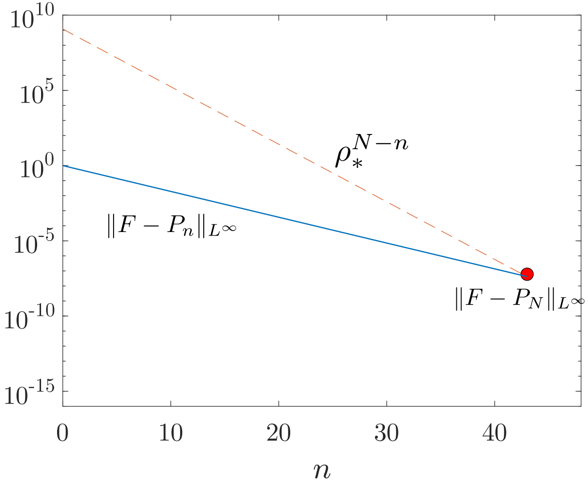

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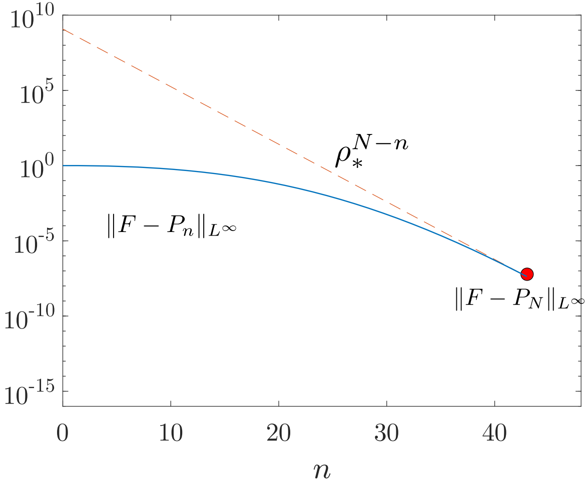

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Recall that

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Therefore,

$$\|a^{(N)}\|_2 \lesssim \rho_*^{N}\|F - P_N\|_{L^\infty(\Gamma)} \cdot N.$$

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We've shown that $\|a^{(N)}\|_2 \lesssim N\rho_*^N \|F - P_N\|_{L^\infty(\Gamma)}$ in this case.

When $\|(V^{(N)})^{-1}\|_2 \lesssim \frac{1}{u}$, the monomial approximation error satisfies

$$\|F - \widehat{P}_N\|_{L^\infty(\Gamma)} \lesssim \|F - P_N\|_{L^\infty(\Gamma)} + u \cdot \|a^{(N)}\|_2$$

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- When $N\rho_*^N \leq \frac{1}{u}$, we have that $\|F - \widehat{P}_N\|_{L^\infty(\Gamma)} \lesssim 2\|F - P_N\|_{L^\infty(\Gamma)}$.
- Recall that $\|(V^{(N)})^{-1}\|_2 \approx \rho_*^N$. So $N\rho_*^N \leq \frac{1}{u}$ generally holds when $\|(V^{(N)})^{-1}\|_2 \leq \frac{1}{u}$.

Implications: stagnation of convergence

We've shown that if $\|F - P_n\|_{L^\infty(\Gamma)}$

- decays at a rate **faster** than ρ_*^{-n},
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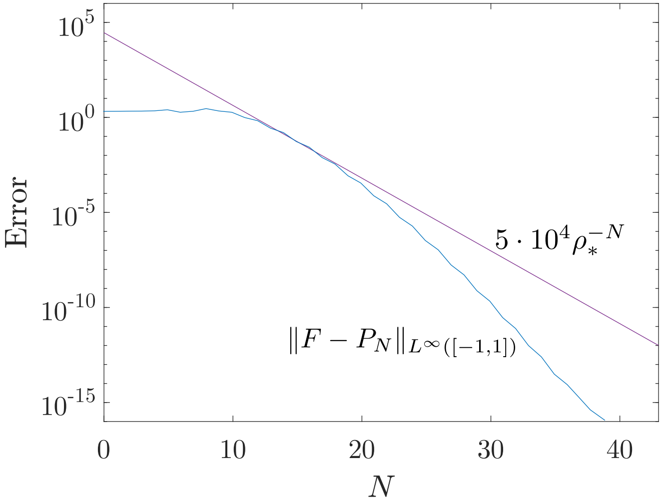
The only way for stagnation to happen before the order reaches the threshold is that, $\|F - P_n\|_{L^\infty(\Gamma)}$ first decays at a rate **slower** than ρ_*^{-n}, then starts to decay at a rate **faster** than ρ_*^{-n}.

Examples: stagnation of convergence

$$F(x) = \cos(12x + 1)$$



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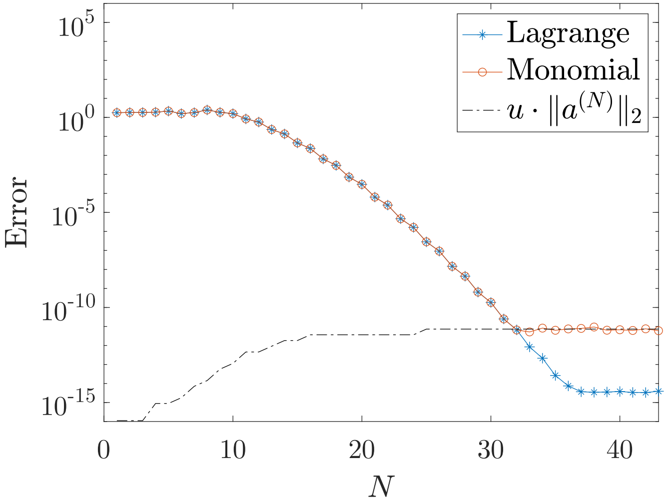
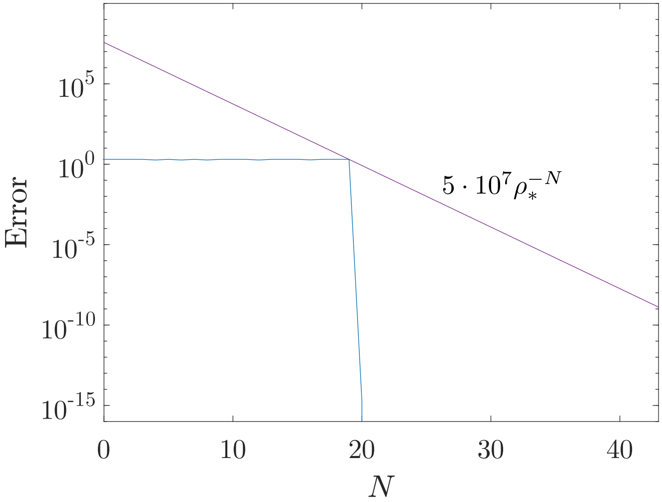


Examples

$$F(x) = T_{20}(x)$$



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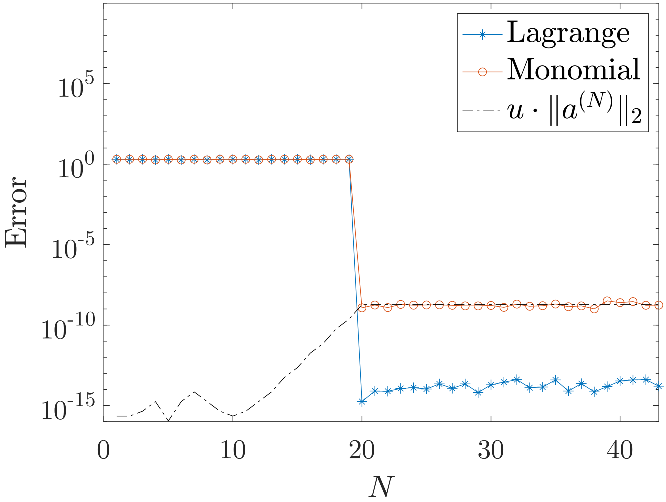

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- In practice, the interpolation error typically doesn't drop like crazy after decaying slowly.
- So stagnation of convergence typically only occurs when N is close to the threshold value.

How restrictive is the monomial basis?

- Extremely high-order interpolation is impossible due to the precondition $\|(V^{(N)})^{-1}\|_2 \lesssim \frac{1}{u}$.
- So **global** interpolation won't work.

How restrictive is the monomial basis?

On the other hand, **piecewise** polynomial interpolation in the monomial basis over a partition of Γ can be carried out stably, provided that

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Rapid evaluations (short expansion, Estrin's scheme).

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- $\mathcal{O}(N^2)$ algorithms exist (could be less backward stable).

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- The threshold ≈ 37 when Γ is a parabola.
- See our paper for experiments.

Generalization to higher dimensions

In 2-D, the Vandermonde matrix looks like

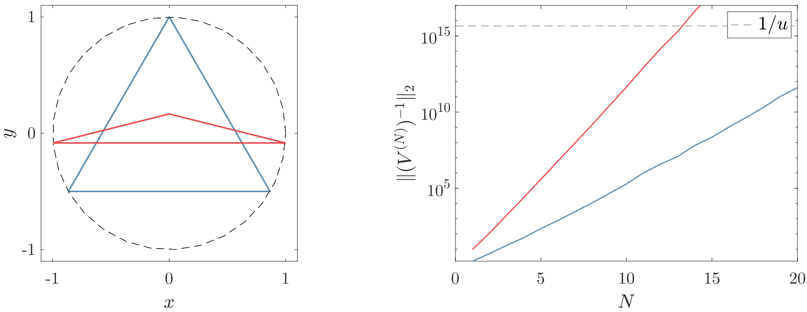
$$V^{(N)} := \begin{pmatrix} 1 & x_1 & y_1 & x_1^2 & x_1 y_1 & \cdots & y_1^N \\ 1 & x_2 & y_2 & x_2^2 & x_2 y_2 & \cdots & y_2^N \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{\widetilde{N}} & y_{\widetilde{N}} & x_{\widetilde{N}}^2 & x_{\widetilde{N}} y_{\widetilde{N}} & \cdots & y_{\widetilde{N}}^N \end{pmatrix},$$

where $\widetilde{N}$ is the dimensionality of bivariate polynomials of order up to N.

Collocation points with relatively small Lebesgue constants have been constructed (Vioreanu & Rokhlin 2014).

The theory of monomial approximation is essentially same as 1-D.

Numerical experiments



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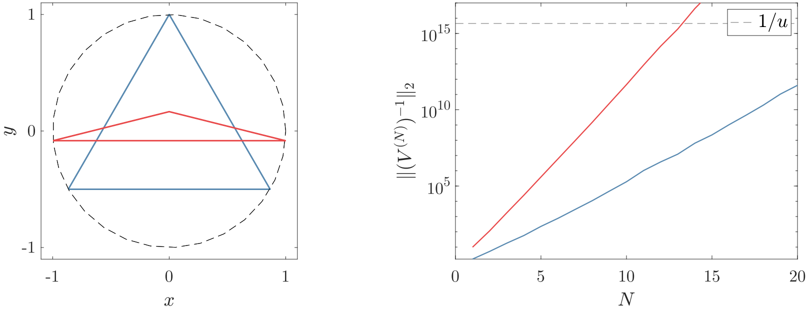
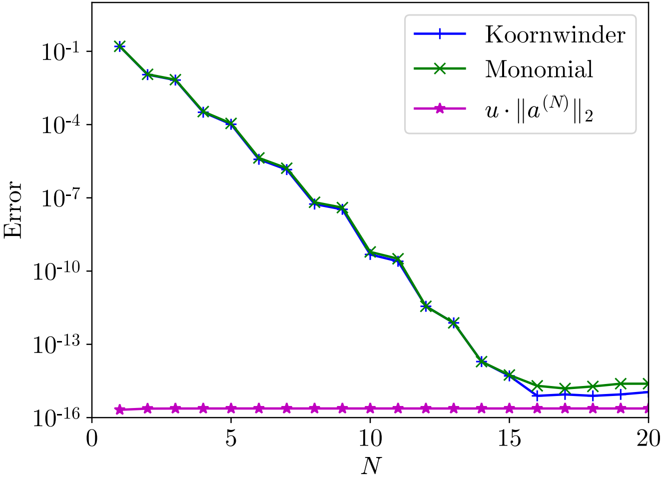


I'll show some experiments that compares the monomial basis with the Koornwinder polynomial basis over the blue triangle.

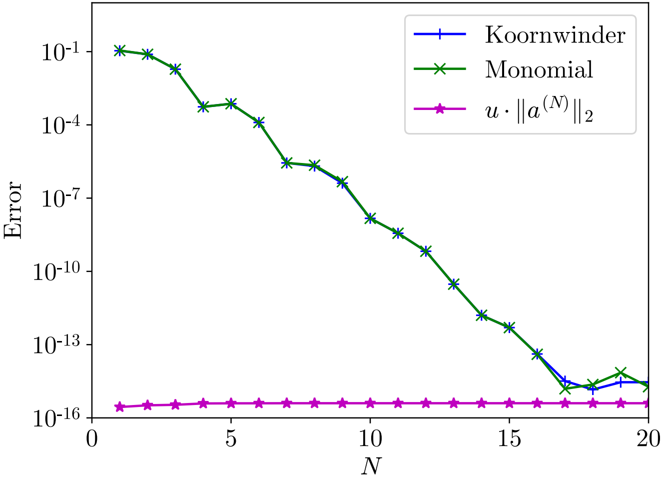
Numerical experiments

$$F(x, y) = e^{-(x^2+y^2)/4}$$



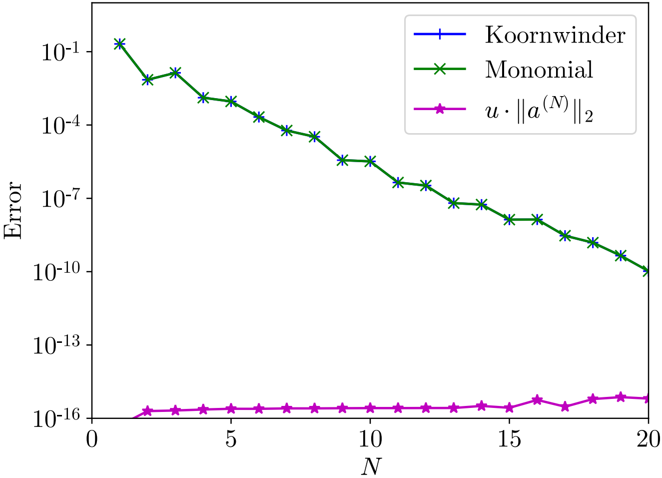
Numerical experiments

$$F(x, y) = \sin(xy/2 + x + y)$$



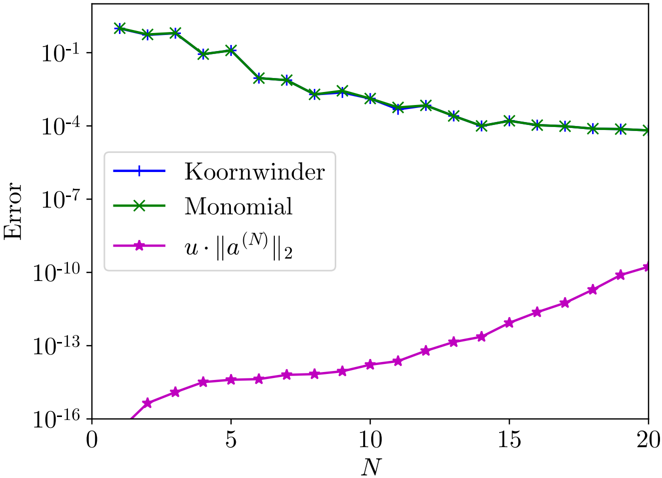
Numerical experiments

$$F(x, y) = \arctan(x) \cdot \arctan(y)$$



Numerical experiments

$$F(x, y) = |x + y|^{5.5}$$



Teaser: Newtonian potential evaluation

Given an irregular domain $\Omega \subset \mathbb{R}^2$ and a function $F : \Omega \to \mathbb{R}$, we're interested in calculating the Newtonian potential

$$u(x) = \iint_{\Omega} \log(\|x - y\|) F(y) \, \mathrm{d}y.$$

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Z. Shen and K. Serkh. "*Rapid evaluation of Newtonian potentials on planar domains.*" arXiv:2208.10443 (2022).

Numerical experiments

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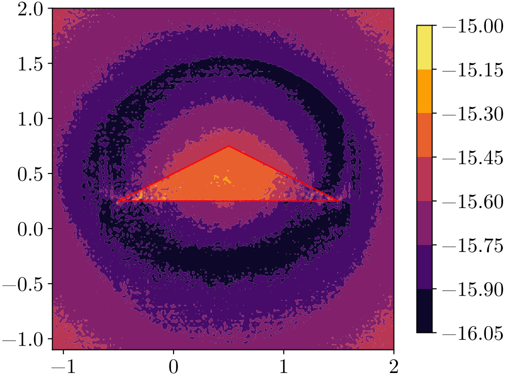

Two orders of magnitude faster than adaptive integration.

Conclusions

Polynomial interpolation in the monomial basis is a valuable tool to have in the numerical toolbox.

Conclusions

An interactive demo:

https://uoft.me/monomial

Paper & slides are available on my personal website (`https://zewenshen.github.io`).

Bonus: what happens when the order $>$ the threshold?

$\cos(12x + 1)$, MATLAB's backslash

Bonus: what happens when the order > the threshold?