Is polynomial interpolation in the monomial basis unstable?

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Oscillatory integrals

Let $\omega \in \mathbb{R}$, and let $F : [-1,1] \to \mathbb{R}$ be a smooth function.

$$\int_{-1}^{1} e^{i\omega x} F(x)\,\mathrm{d}x$$

It requires $\mathcal{O}(\omega)$ operations to compute this integral by using standard quadrature rules.

$$\int_{-1}^{1} e^{i\omega x}\,\mathrm{d}x = \frac{1}{i\omega}(e^{i\omega} - e^{-i\omega})$$

$$\int_{-1}^{1} e^{i\omega x} x^{k+1}\,\mathrm{d}x = \frac{1}{i\omega}\Big(e^{i\omega} + (-1)^k e^{-i\omega} - (k+1)\int_{-1}^{1} e^{i\omega x} x^k\,\mathrm{d}x\Big)$$

If the function F is approximated by a monomial expansion, the computational cost is independent of ω.

Singular integrals (Helsing & Ojala 2008)

Let $\Gamma \subset \mathbb{C}$ be a smooth curve. Given an analytic function $F : \Gamma \to \mathbb{C}$ and a point $\xi \in \mathbb{C}$ close to Γ,

$$\int_\Gamma \frac{F(z)}{z - \xi} \, \mathrm{d}z$$

is costly to evaluate using adaptive integration.

$$\int_\Gamma \frac{1}{z - \xi} \, \mathrm{d}z = \log(z_1 - \xi) - \log(z_0 - \xi) + 2\pi i \mathcal{N}_\xi$$

$$\int_\Gamma \frac{z^k}{z - \xi} \, \mathrm{d}z = \frac{1 - (-1)^{k-1}}{k - 1} + \xi \int_\Gamma \frac{z^{k-1}}{z - x} \, \mathrm{d}z$$

Hadamard finite-part integral

Given $\nu \in \mathbb{R}$ and $M \in \mathbb{N}_{\geq 0}$, we're interested in the calculation of the Hadamard finite-part integral

$$\mathrm{f.p.} \int_0^1 x^\nu \log^m(x) \cdot F(x) \, \mathrm{d}x.$$

$$\mathrm{f.p.} \int_0^1 x^\nu \log^m(x) \cdot x^k \, \mathrm{d}x = \frac{(-1)^m m!}{(\nu + k + 1)^{m+1}}.$$

Particular solution to Poisson's equation

Given a domain $\Omega \in \mathbb{R}^2$, and a function $F : \Omega \to \mathbb{R}$, find a solution to

$$\nabla^2 u = F.$$

For all $m \geq 0$,

$$\nabla^{-2}[x^m] = \frac{1}{(m+1)(m+2)} x^{m+2}, \quad \nabla^{-2}[x^m y] = \frac{1}{(m+1)(m+2)} x^{m+2} y.$$

For all $n \geq 0$,

$$\nabla^{-2}[y^n] = \frac{1}{(n+1)(n+2)} y^{n+2}, \quad \nabla^{-2}[x y^n] = \frac{1}{(n+1)(n+2)} x y^{n+2}.$$

When $F = x^m y^n$ for some $m \geq 2$, $n \geq 2$,

$$\begin{aligned}
\nabla^{-2}[x^m y^n] &= \frac{x^{m+2} y^n}{(m+2)(m+1)} - \frac{n(n-1)}{(m+2)(m+1)} \nabla^{-2}[x^{m+2} y^{n-2}] \\
&= \frac{x^m y^{n+2}}{(n+2)(n+1)} - \frac{m(m-1)}{(n+2)(n+1)} \nabla^{-2}[x^{m-2} y^{n+2}]
\end{aligned}$$

Motivations

- How to approximate functions by monomials?
- General attitude has remained skeptical.

Polynomial interpolation

Polynomials are powerful tools for approximating functions.

Definition

Given a function $F : [-1, 1] \to \mathbb{C}$, the Nth degree interpolating polynomial P_N of F satisfies $P_N(x_j) = F(x_j)$, for a set of $(N + 1)$ distinct collocation points $\{x_j\}_{j=0,1,\ldots,N}$.

The choice of collocation points is important for good approximation quality.

Polynomial interpolation

- Polynomial interpolation can be viewed as a linear operator from $\big(F(x_0), F(x_1), \ldots, F(x_N)\big)$ to the interpolating polynomial P_N.
- The Lebesgue constant Λ_N for $\{x_j\}_{j=0,1,\ldots,N}$ is the $(\ell^\infty, L^\infty([-1,1]))$ norm of this linear operator.
- $\|F - P_N\|_{L^\infty([-1,1])} \leq (1 + \Lambda_N) \cdot \inf_{p \in \mathcal{P}_N} \|F - p\|_{L^\infty([-1,1])}$.

We'll choose collocation points with small Λ_N.

Polynomial interpolation in finite precision

To compute P_N on a computer, we first choose a polynomial basis $\{\phi_k\}_k$

$$P_N(x) = \sum_{k=0}^{N} a_k \phi_k(x)$$

$$\begin{pmatrix} \phi_0(x_0) & \phi_1(x_0) & \phi_2(x_0) & \cdots & \phi_N(x_0) \\ \phi_0(x_1) & \phi_1(x_1) & \phi_2(x_1) & \cdots & \phi_N(x_1) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ \phi_0(x_N) & \phi_1(x_N) & \phi_2(x_N) & \cdots & \phi_N(x_N) \end{pmatrix} \begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_N \end{pmatrix} = \begin{pmatrix} F(x_0) \\ F(x_1) \\ \vdots \\ F(x_N) \end{pmatrix}.$$

Why does the choice of basis matter?

- Condition number
- Time complexity

The standard choices:

- Lagrange polynomials.
- Orthogonal polynomials (Chebyshev, Legendre, etc).

Polynomial interpolation in the monomial basis

What about expressing P_N in the monomial basis?

$$P_N(x) = \sum_{k=0}^{N} a_k x^k$$

The previous linear system becomes

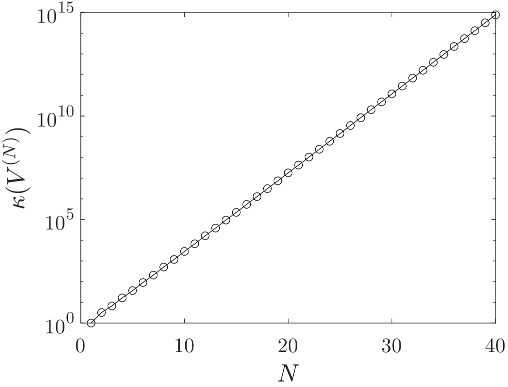
$$\underbrace{\begin{pmatrix} 1 & x_0 & x_0^2 & \cdots & x_0^N \\ 1 & x_1 & x_1^2 & \cdots & x_1^N \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_N & x_N^2 & \cdots & x_N^N \end{pmatrix}}_{V^{(N)}} \underbrace{\begin{pmatrix} a_0 \\ a_1 \\ \vdots \\ a_N \end{pmatrix}}_{a^{(N)}} = \underbrace{\begin{pmatrix} F(x_0) \\ F(x_1) \\ \vdots \\ F(x_N) \end{pmatrix}}_{f^{(N)}}.$$

$V^{(N)}$ is known as a Vandermonde matrix.

Monomial basis is ill-conditioned

Given any set of real collocation points, $\kappa(V^{(N)})$ grows at least exponentially fast.

Example: when the Chebyshev points are used for collocation:



Numerical experiments

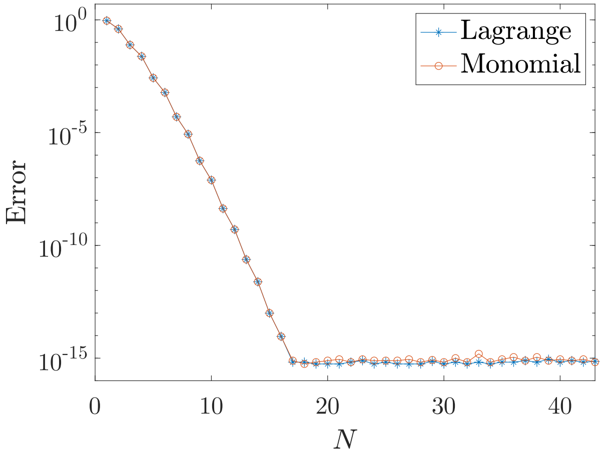
Let's run some experiments. The following quantities will be reported.

- $\|F - \widehat{P}_N\|_{L^\infty([-1,1])}$: Monomial approximation error.
  Denoted by the label "monomial".
- $\|F - P_N\|_{L^\infty([-1,1])}$: Exact polynomial interpolation error, estimated using the Barycentric interpolation formula.
  Denoted by the label "Lagrange".

Chebyshev points are used for collocation.

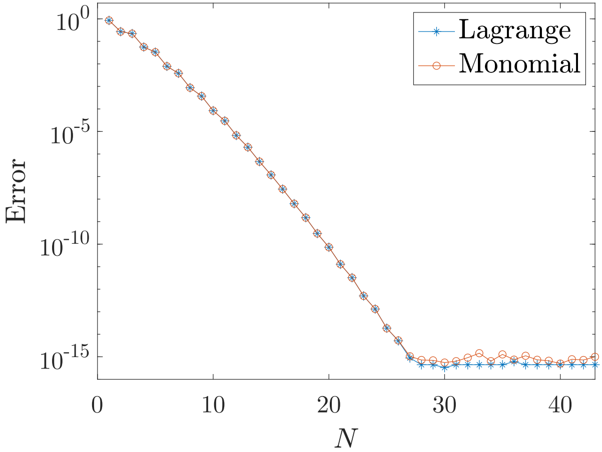
Numerical experiments

$$F(x) = \cos(2x + 1)$$



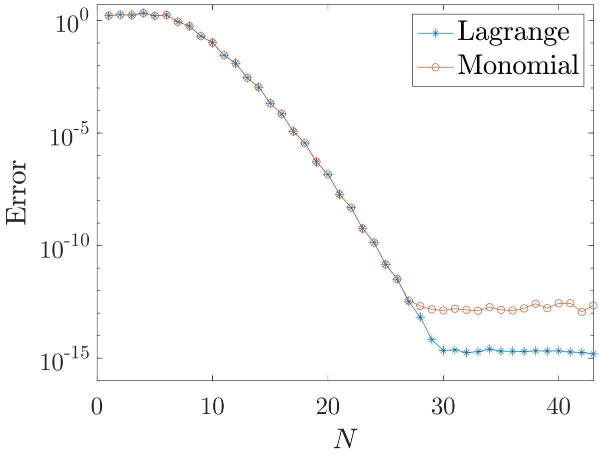
Numerical experiments

$$F(x) = e^{-2(x+0.1)^2}$$



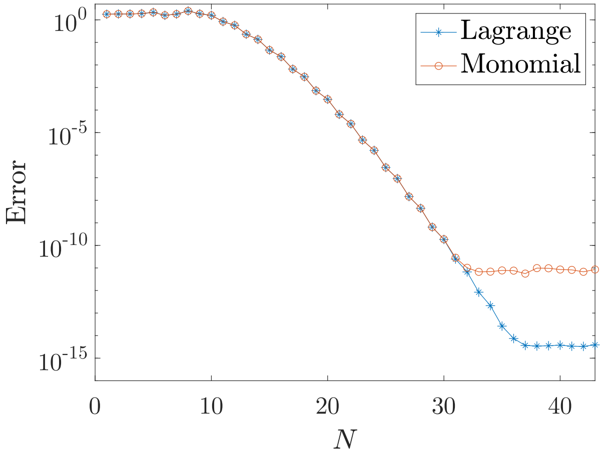
Numerical experiments

$$F(x) = \cos(8x + 1)$$



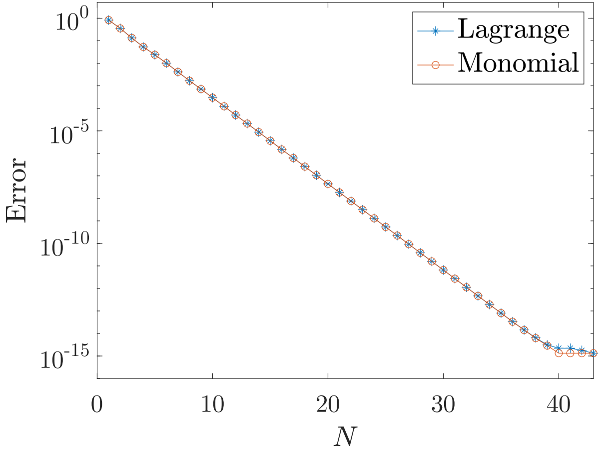
Numerical experiments

$$F(x) = \cos(12x + 1)$$



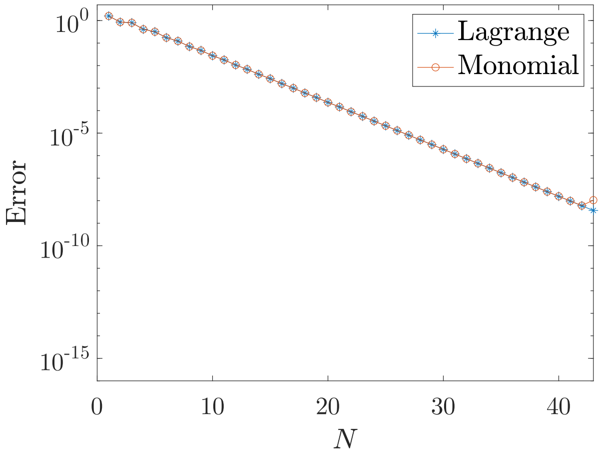
Numerical experiments

$$F(x) = \frac{1}{x-\sqrt{2}}$$



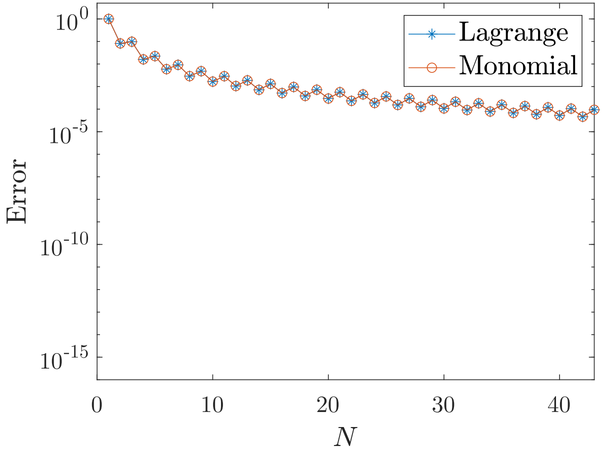
Numerical experiments

$$F(x) = \frac{1}{x - 0.5i}$$



Numerical experiments

$$F(x) = |x|^{2.5}$$



Reflections

- Polynomial interpolation in the monomial basis is **not** as unstable as it appears.
- The same thing happens when the domain is not $[-1, 1]$ (say, a smooth simple arc $\Gamma \subset \mathbb{C}$, a triangle in $\mathbb{R}^2$).
- The observation that "the monomials can approximate sufficiently smooth functions to high accuracy" dates back to ≥ 25 years ago.
- Not very widely known. Not fully understood. Not the complete story.

The monomial basis is not too different from a well-conditioned polynomial basis for interpolation, provided that $\kappa(V^{(N)}) \leq \frac{1}{u}$.

Before I explain why, I'll present my favorite application.

Monomial as the default choice for interpolation

Well-conditioned polynomial bases for irregular domains are generally either unknown or complicated.

- What are orthogonal polynomials over an arbitrary curved triangle?
- Orthogonal polynomials over a standard simplex:

$$K_{mn}(x,y) = (1-x)^m \cdot P_{n-m}^{(2m+1,0)}(2x-1) \cdot P_m\left(\frac{2y}{1-x} - 1\right).$$

On the other hand, this is what a monomial look like:

$$x^m y^n.$$

Works for any domain. Much more handy. Much cheaper to evaluate.

Assumptions

In the rest of the talk, here's a list of our assumptions:

- The domain of approximation $\Gamma \subset \mathbb{C}$ can be an arbitrary smooth simple arc.
- Γ is inside the unit disk D_1 centered at the origin.
- The Lebesgue constant Λ_N of the collocation points is small.

Feel free to consider $\Gamma = [-1, 1]$ with Chebyshev points.

Rethinking interpolation

Huge condition number of Vandermonde matrices

$$\Longrightarrow$$

extremely inaccurate monomial coefficients

Do we care about the accuracy of the computed monomial coefficients?

What's really important is the backward error, i.e.,

$$\|V^{(N)}\widehat{a}^{(N)} - f^{(N)}\|_2,$$

of the numerical solution $\widehat{a}^{(N)}$ to the Vandermonde system $V^{(N)}a^{(N)} = f^{(N)}$.

Rethinking interpolation

The difference between the exact interpolating polynomial P_N and the computed monomial expansion $\widehat{P}_N$ satisfies

$$\|P_N - \widehat{P}_N\|_{L^\infty(\Gamma)} \leq \Lambda_N \|V^{(N)}\widehat{a}^{(N)} - f^{(N)}\|_2.$$

How large will the backward error be?

Backward stable linear system solver

When a backward stable linear system solver is used to solve the Vandermonde system $V^{(N)}a^{(N)} = f^{(N)}$, the numerical solution $\widehat{a}^{(N)}$ is the exact solution to

$$\left(V^{(N)} + \delta V^{(N)}\right)\widehat{a}^{(N)} = f^{(N)},$$

for some $\delta V^{(N)} \in \mathbb{C}^{(N+1)\times(N+1)}$ that satisfies

$$\|\delta V^{(N)}\|_2 \leq u \cdot \gamma_N,$$

where u denotes machine epsilon and $\gamma_N = \mathcal{O}(\|V^{(N)}\|_2)$.

Remark: $\|V^{(N)}\|_2$ is small when $\Gamma \subset D_1$. When MATLAB's backslash is used, we observe that $\gamma_N \lesssim 1$ for at least $N \leq 100$.

It follows that

$$\|V^{(N)}\widehat{a}^{(N)} - f^{(N)}\|_2 = \|\delta V^{(N)}\widehat{a}^{(N)}\|_2 \leq u \cdot \gamma_N \|\widehat{a}^{(N)}\|_2.$$

A priori estimate

What do we know about $\|\widehat{a}^{(N)}\|_2$?

$$\big(V^{(N)} + \delta V^{(N)}\big)\widehat{a}^{(N)} = f^{(N)} \iff \Big(I + \big(V^{(N)}\big)^{-1}\big(\delta V^{(N)}\big)\Big)\widehat{a}^{(N)} = a^{(N)}.$$

Recall that $\|\delta V^{(N)}\|_2 \leq u \cdot \gamma_N$. If $\|\big(V^{(N)}\big)^{-1}\|_2 \leq \frac{1}{2u \cdot \gamma_N}$, we have that

$$\big\|\big(V^{(N)}\big)^{-1}\big(\delta V^{(N)}\big)\big\|_2 \leq \big\|\big(V^{(N)}\big)^{-1}\big\|_2 \cdot \big\|\delta V^{(N)}\big\|_2 \leq \frac{1}{2}.$$

It follows from the geometric series theorem that $\|\widehat{a}^{(N)}\|_2 \leq 2\|a^{(N)}\|_2$.

Monomial approximation error

We've shown that

$$\|P_N - \widehat{P}_N\|_{L^\infty(\Gamma)} \leq \Lambda_N \|V^{(N)}\widehat{a}^{(N)} - f^{(N)}\|_2,$$

and that

$$\|(V^{(N)})^{-1}\|_2 \leq \frac{1}{2u \cdot \gamma_N} \implies \left\|V^{(N)}\widehat{a}^{(N)} - f^{(N)}\right\|_2 \leq 2u \cdot \gamma_N \|a^{(N)}\|_2.$$

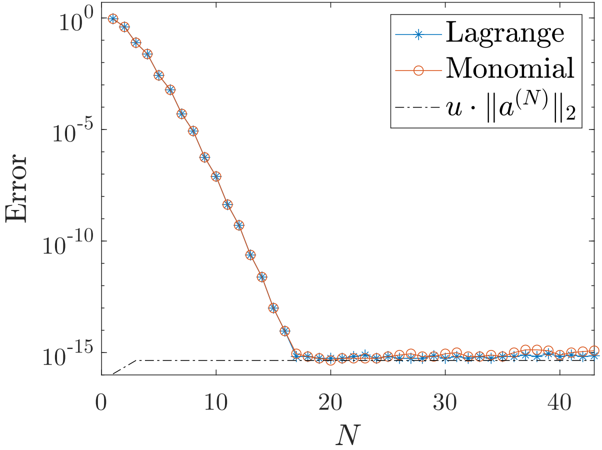
Assume that $\|(V^{(N)})^{-1}\|_2 \leq \frac{1}{2u \cdot \gamma_N}$. By the triangle inequality, the monomial approximation error satisfies

$$\begin{aligned} \|F - \widehat{P}_N\|_{L^\infty(\Gamma)} &\leq \|F - P_N\|_{L^\infty(\Gamma)} + \|P_N - \widehat{P}_N\|_{L^\infty(\Gamma)} \\ &\leq \|F - P_N\|_{L^\infty(\Gamma)} + \Lambda_N \|V^{(N)}\widehat{a}^{(N)} - f^{(N)}\|_2 \\ &\leq \|F - P_N\|_{L^\infty(\Gamma)} + 2u \cdot \gamma_N \Lambda_N \|a^{(N)}\|_2. \end{aligned}$$

- Extra additive error term $\approx u \cdot \|a^{(N)}\|_2$.

Numerical experiments

$$F(x) = \cos(2x + 1)$$



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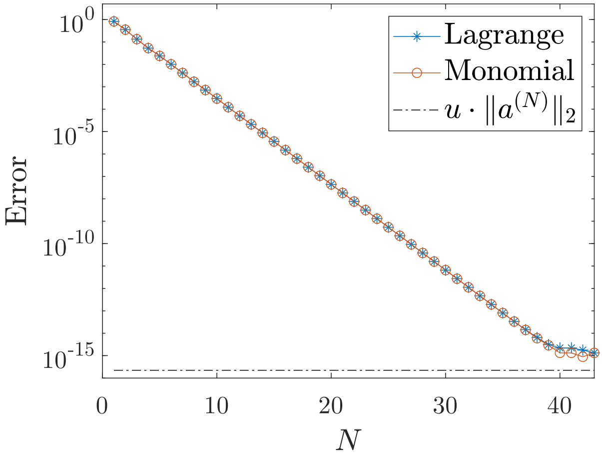

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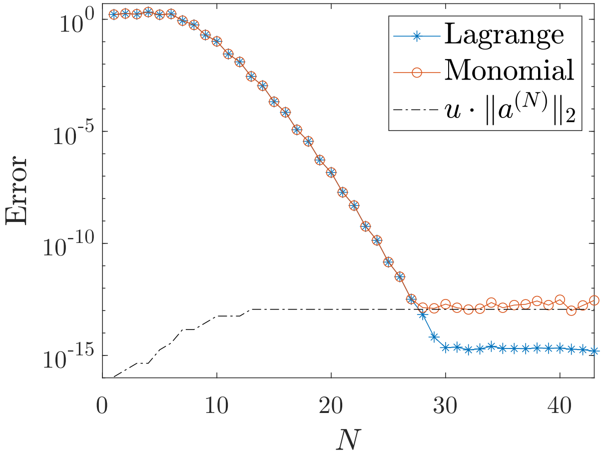

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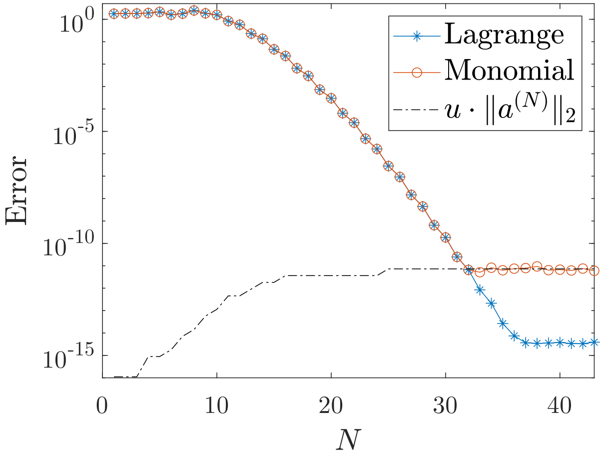

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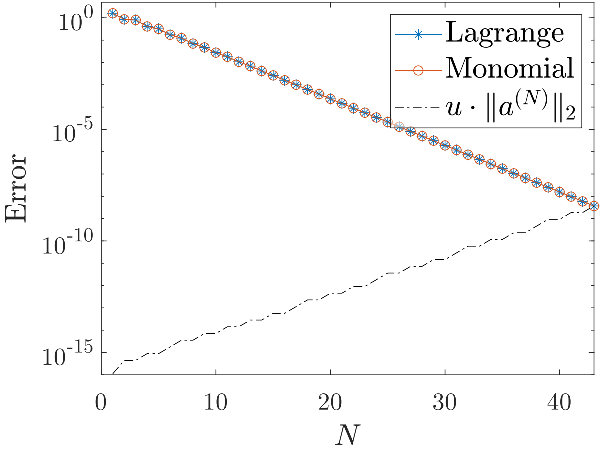

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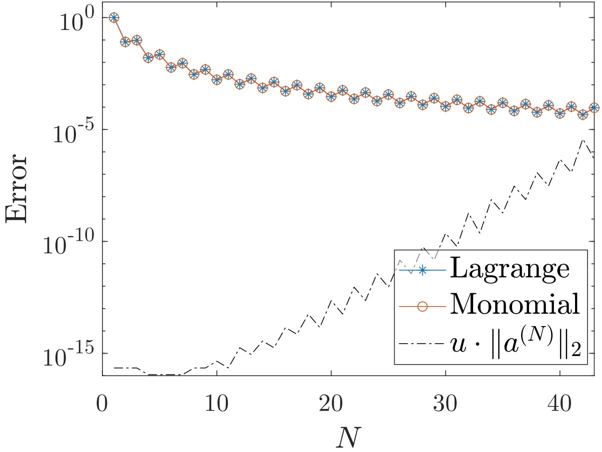


Story so far

- We can now explain these experiments, but many things are still unclear.
- For example, when will the extra error (i.e., $u \cdot \|a^{(N)}\|_2$) be small?
- This requires an a priori estimate for the growth of $\|a^{(N)}\|_2$.

Remark: The literature sometimes cites backward stability as the only justification for the use of the monomial basis, which is somewhat misguided.

Monomial coefficients of P_N

Lemma

Let $P_N : \mathbb{C} \to \mathbb{C}$ be a polynomial of degree N, where $P_N(z) = \sum_{k=0}^{N} a_k z^k$ for some $a_0, a_1, \dots, a_N \in \mathbb{C}$. The 2-norm of the coefficient vector $a^{(N)} := (a_0, a_1, \dots, a_N)^T$ satisfies

$$\|a^{(N)}\|_2 \leq \|P_N\|_{L^\infty(\partial D_1)},$$

where D_1 denotes the open unit disk centered at the origin.

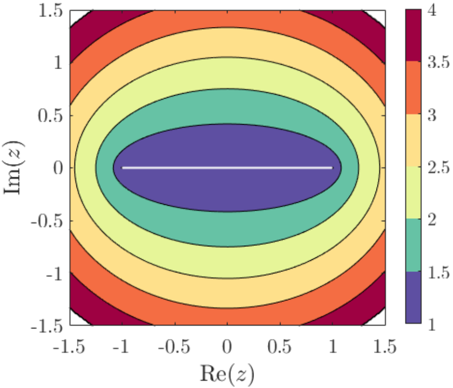
Proof.

Observe that $P_N(e^{i\theta}) = \sum_{k=0}^{N} a_k e^{ik\theta}$. Thus, by Parseval's identity, we have that $\|a^{(N)}\|_2 = \left(\frac{1}{2\pi}\int_0^{2\pi} |P_N(e^{i\theta})|^2 \, \mathrm{d}\theta\right)^{1/2} \leq \|P_N\|_{L^\infty(\partial D_1)}$. $\square$

- Very tight bound, but relies on knowledge of $\|P_N\|_{L^\infty(\partial D_1)}$.

Bernstein ellipse

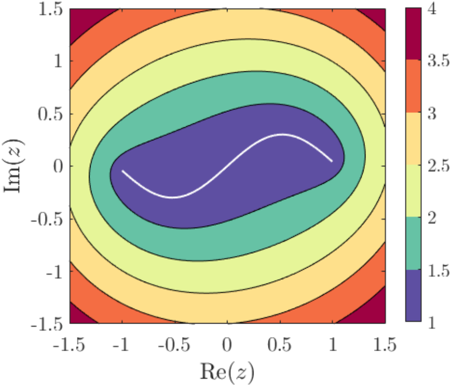
Given $\rho > 1$, the Bernstein ellipse E_ρ for $\Gamma = [-1, 1]$ is the image of a circle centered at the origin with radius ρ under the mapping $\frac{1}{2}(z + \frac{1}{z})$.



- The larger ρ, the bigger the ellipse.
- We let E_ρ^o denote the open region bounded by E_ρ.

Generalization of Bernstein ellipse

The concept of the Bernstein ellipse can be generalized to an arbitrary smooth simple arc $\Gamma \subset \mathbb{C}$.



Bernstein's inequality

Lemma (Walsh 1935)

Let Γ be a smooth simple arc in the complex plane, and let E_ρ^o be the region corresponding to Γ with some parameter $\rho > 1$. Then, the L^∞ norm of any polynomial P_N of degree N over E_ρ^o satisfies

$$\|P_N\|_{L^\infty(E_\rho^o)} \leq \rho^N \|P_N\|_{L^\infty(\Gamma)}.$$

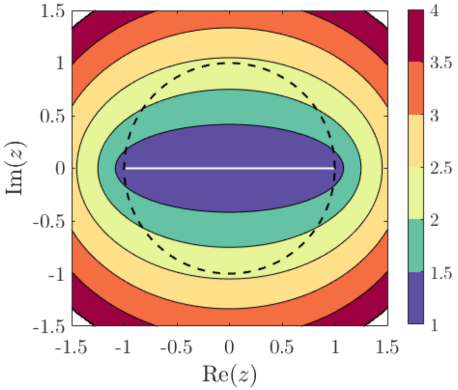
Define $\rho_* = \min\{\rho > 1 : D_1 \subset E_\rho^o\}$. Then,

$$\|a^{(N)}\|_2 \leq \|P_N\|_{L^\infty(\partial D_1)} \leq \|P_N\|_{L^\infty(E_{\rho_*}^o)} \leq \rho_*^N \|P_N\|_{L^\infty(\Gamma)}.$$

Examples of ρ_*

Example

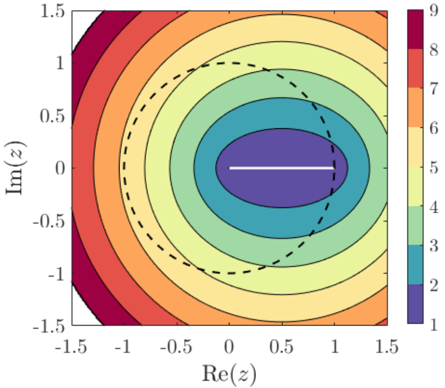
When $\Gamma = [-1, 1]$, $\rho_* = 1 + \sqrt{2} \approx 2.4$



Examples of ρ_*

Example

When $\Gamma = [0, 1]$, $\rho_* = 3 + 2\sqrt{2} \approx 5.8$



An upper bound for $\|(V^{(N)})^{-1}\|_2$

We've shown that

$$\frac{\|a^{(N)}\|_2}{\|P_N\|_{L^\infty(\Gamma)}} \leq \rho_*^N.$$

Also note that

$$\|(V^{(N)})^{-1}\|_2 = \sup_{f^{(N)}\neq 0}\left\{\frac{\|(V^{(N)})^{-1}f^{(N)}\|_2}{\|f^{(N)}\|_2}\right\} = \sup_{f^{(N)}\neq 0}\left\{\frac{\|a^{(N)}\|_2}{\|f^{(N)}\|_2}\right\}.$$

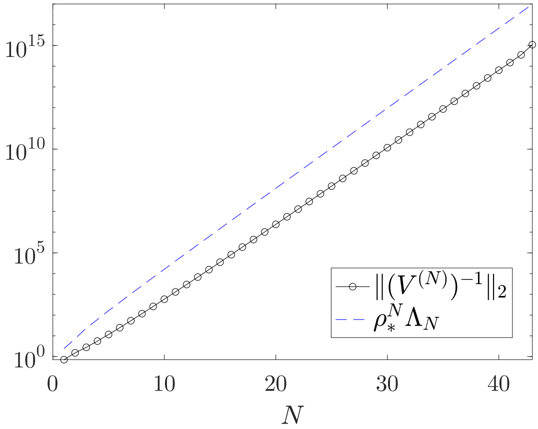
Theorem (Shen & Serkh (2022))

Suppose that $V^{(N)} \in \mathbb{C}^{(N+1)\times(N+1)}$ is a Vandermonde matrix with $(N+1)$ distinct collocation points over $\Gamma \subset \mathbb{C}$. Then,

$$\|(V^{(N)})^{-1}\|_2 \leq \rho_*^N \Lambda_N.$$

An upper bound for $\|(V^{(N)})^{-1}\|_2$

$\Gamma = [-1, 1]$, $\rho_* = 1 + \sqrt{2}$, Chebyshev points



An upper bound for $\|(V^{(N)})^{-1}\|_2$

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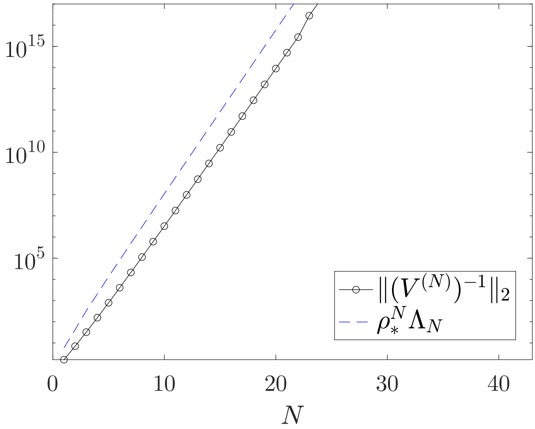


An upper bound for $\|a^{(N)}\|_2$

Theorem (Shen & Serkh (2022))

Suppose that there exists a finite sequence of polynomials $\{Q_n\}_{n=0,1,\dots,N}$*, where* Q_n *has degree* n*, which satisfies*

$$\|F - Q_n\|_{L^\infty(\Gamma)} \le C\rho^{-n}, \quad 0 \le n \le N,$$

for some constants $\rho > 1$ *and* $C \ge 0$*. The 2-norm of the monomial coefficient vector of the* N*th degree interpolating polynomial* P_N *of* F *satisfies*

$$\|a^{(N)}\|_2 \le \|F\|_{L^\infty(\Gamma)} + C\left(\Lambda_N\left(\frac{\rho_*}{\rho}\right)^N + 2\rho_* \sum_{j=0}^{N-1}\left(\frac{\rho_*}{\rho}\right)^j + 1\right).$$

A simplified upper bound for $\|a^{(N)}\|_2$

We fix the variable ρ to be ρ_*. The previous theorem becomes:

$$\|F - Q_n\|_{L^\infty(\Gamma)} \leq C\rho_*^{-n}, \quad 0 \leq n \leq N,$$

$$\Longrightarrow$$

$$\|a^{(N)}\|_2 \leq \|F\|_{L^\infty(\Gamma)} + C\left(\Lambda_N\left(\frac{\rho_*}{\rho_*}\right)^N + 2\rho_* \sum_{j=0}^{N-1}\left(\frac{\rho_*}{\rho_*}\right)^j + 1\right) \lesssim C \cdot N.$$

In practice, one can take $\{Q_n\}_{n=0,1,\ldots,N}$ to be a finite sequence of interpolating polynomials $\{P_n\}_{n=0,1,\ldots,N}$ of F.

We first deal with the case where the $\|F - P_n\|_{L^\infty(\Gamma)} \lesssim \rho_*^{-n}$.

$$\|a^{(N)}\|_2 \lesssim C \cdot N \approx N.$$

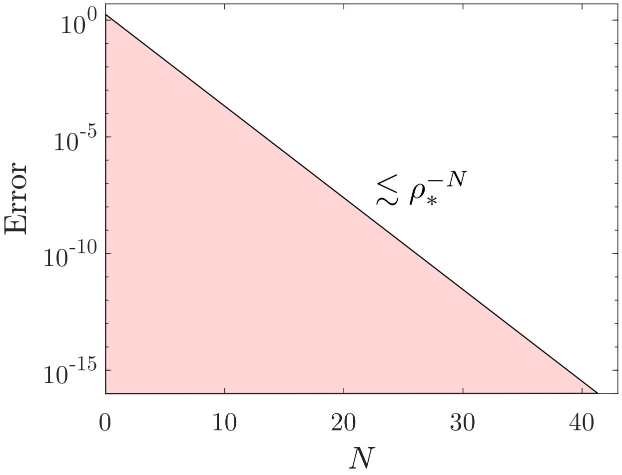
Implications: when $\|F - P_N\|_{L^\infty(\Gamma)}$ decays quickly

Therefore, when $\|(V^{(N)})^{-1}\|_2 \lesssim \frac{1}{u}$, the monomial approximation error satisfies

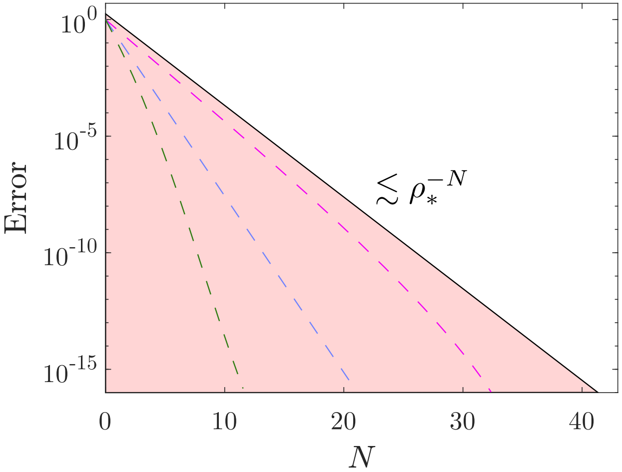
$$\|F - \widehat{P}_N\|_{L^\infty(\Gamma)} \lesssim \|F - P_N\|_{L^\infty(\Gamma)} + u \cdot N.$$

The extra error is around machine epsilon in this case!

Visualization: when $\|F - P_N\|_{L^\infty(\Gamma)}$ decays quickly



Visualization: when $\|F - P_N\|_{L^\infty(\Gamma)}$ decays quickly

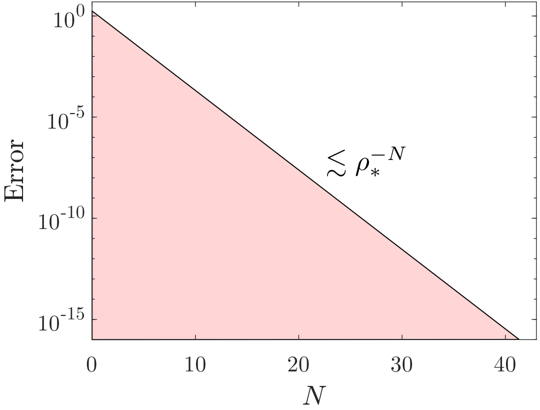


Does the precondition $\|(V^{(N)})^{-1}\|_2 \lesssim \frac{1}{u}$ weaken our result?

Implications: when $\|F - P_N\|_{L^\infty(\Gamma)}$ decays quickly

Recall that $\|(V^{(N)})^{-1}\|_2 \leq \rho_*^N \Lambda_N$.

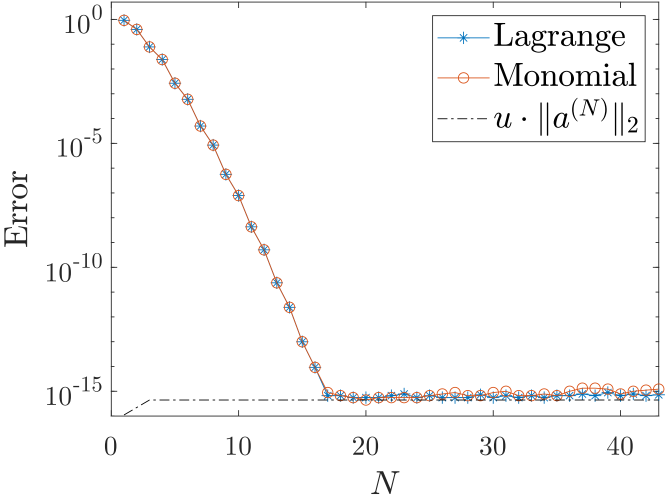
When N satisfies $\rho_*^{-N} = u$, $\|(V^{(N)})^{-1}\|_2 \leq \frac{\Lambda_N}{u} \lesssim \frac{1}{u}$.



The threshold will always be on the right of this pink region.

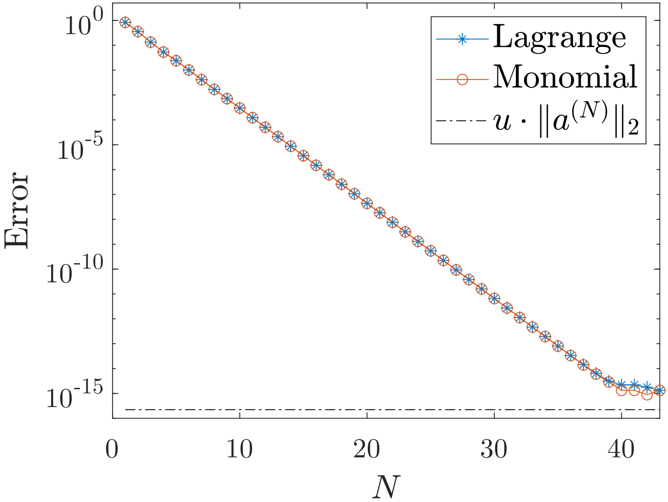
Examples: when $\|F - P_N\|_{L^\infty(\Gamma)}$ decays quickly

$$F(x) = \cos(2x + 1)$$



Examples: when $\|F - P_N\|_{L^\infty(\Gamma)}$ decays quickly

$$F(x) = \frac{1}{x - \sqrt{2}}$$



Implications: when $\|F - P_N\|_{L^\infty(\Gamma)}$ decays slowly

When $\|F - P_n\|_{L^\infty(\Gamma)} \lesssim \rho_*^{-n}$ for $0 \leq n \leq N$,

- the growth of $\|a^{(N)}\|_2$ is suppressed,
- and one loses nothing by using the monomial basis.

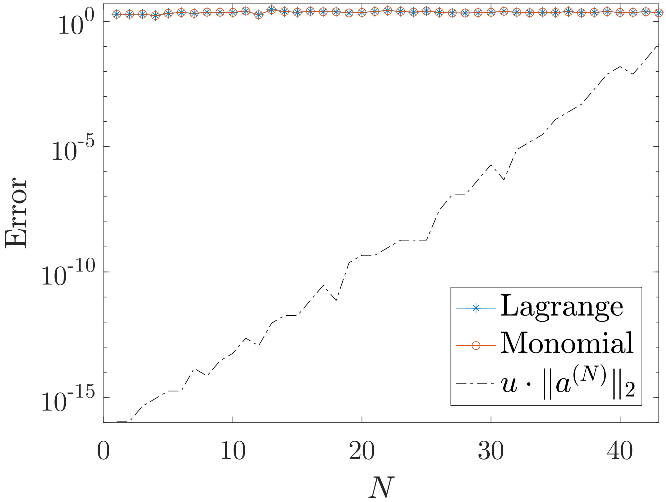
What happens if the polynomial interpolation error decays more slowly?

- $\|a^{(N)}\|_2$ will be larger.
- extra error caused by the monomial basis becomes non-negligible.

Does it matter?

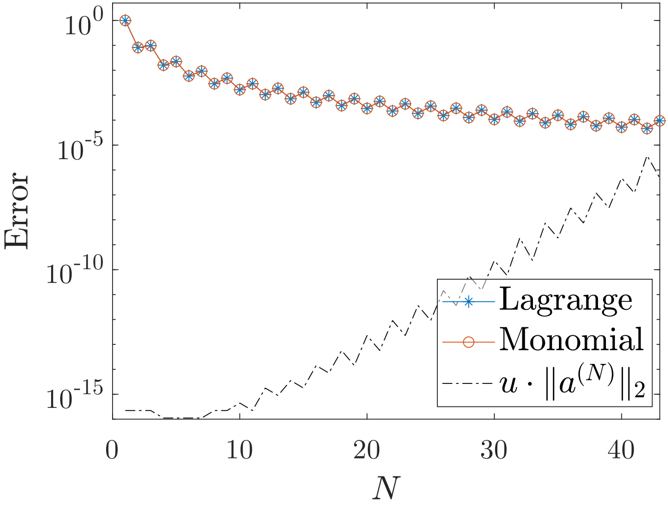
Examples: when $\|F - P_N\|_{L^\infty(\Gamma)}$ decays slowly

$$F(x) = \cos(120x + 1)$$



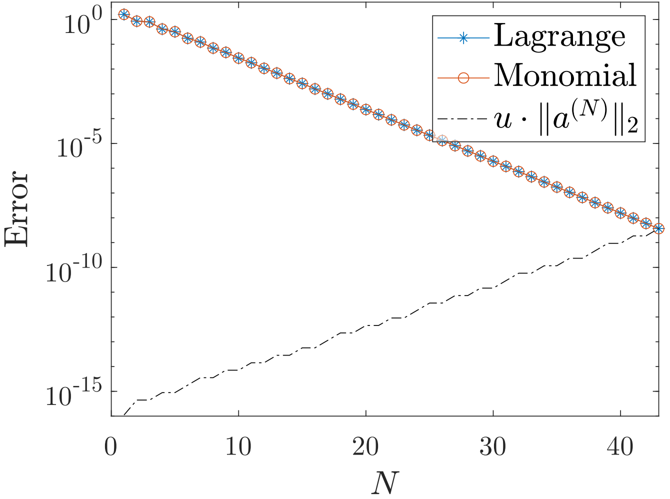
Examples: when $\|F - P_N\|_{L^\infty(\Gamma)}$ decays slowly

$$F(x) = |x|^{5/2}$$



Examples: when $\|F - P_N\|_{L^\infty(\Gamma)}$ decays slowly

$$F(x) = \frac{1}{x - 0.5i}$$



Implications: when $\|F - P_N\|_{L^\infty(\Gamma)}$ decays slowly

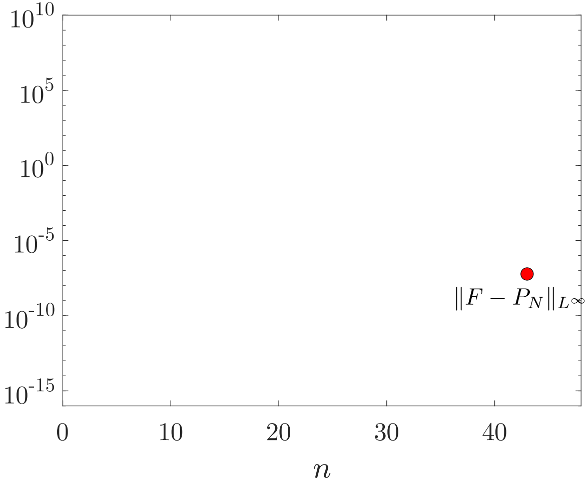
I'll now characterize what we just observed.

Assume that $\|F - P_n\|_{L^\infty(\Gamma)}$ decays to the value $\|F - P_N\|_{L^\infty(\Gamma)}$ at a rate slower than ρ_*^{-n}, i.e.,

$$\|F - P_n\|_{L^\infty(\Gamma)} \leq \rho_*^{N-n} \|F - P_N\|_{L^\infty(\Gamma)}, \quad \text{for } 0 \leq n \leq N.$$

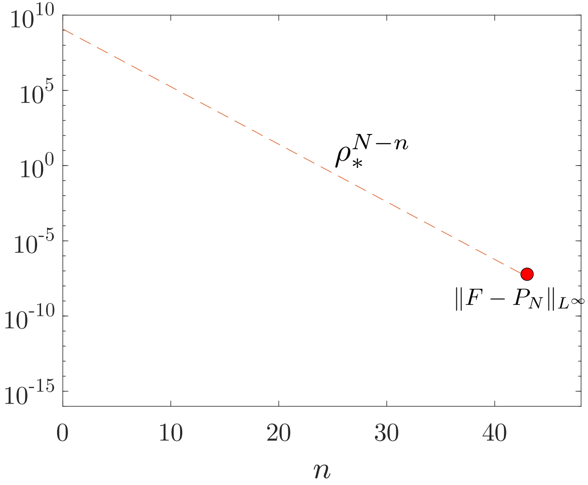
Visualizations: when $\|F - P_N\|_{L^\infty(\Gamma)}$ decays slowly

$$\|F - P_n\|_{L^\infty(\Gamma)} \leq \rho_*^{N-n} \|F - P_N\|_{L^\infty(\Gamma)}, \quad \text{for } 0 \leq n \leq N.$$



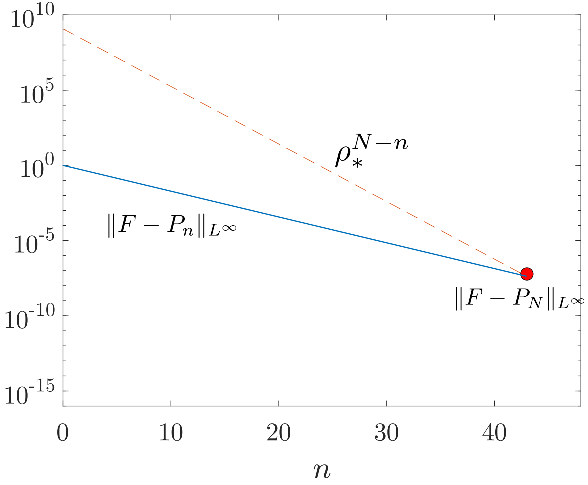
Visualizations: when $\|F - P_N\|_{L^\infty(\Gamma)}$ decays slowly

$$\|F - P_n\|_{L^\infty(\Gamma)} \le \rho_*^{N-n} \|F - P_N\|_{L^\infty(\Gamma)}, \quad \text{for } 0 \le n \le N.$$



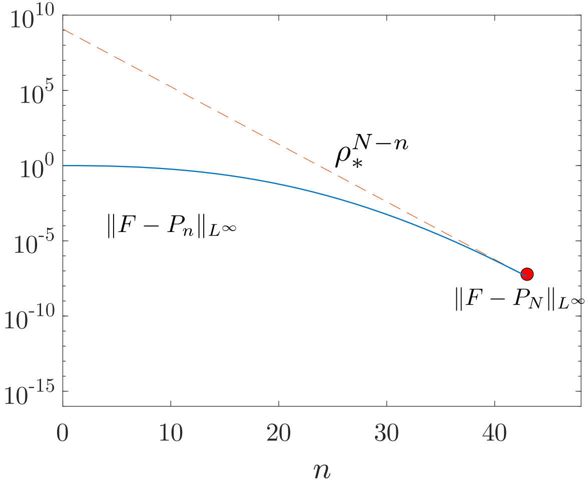
Visualizations: when $\|F - P_N\|_{L^\infty(\Gamma)}$ decays slowly

$$\|F - P_n\|_{L^\infty(\Gamma)} \leq \rho_*^{N-n} \|F - P_N\|_{L^\infty(\Gamma)}, \quad \text{for } 0 \leq n \leq N.$$



Visualizations: when $\|F - P_N\|_{L^\infty(\Gamma)}$ decays slowly

$$\|F - P_n\|_{L^\infty(\Gamma)} \leq \rho_*^{N-n} \|F - P_N\|_{L^\infty(\Gamma)}, \quad \text{for } 0 \leq n \leq N.$$



Implications: when $\|F - P_N\|_{L^\infty(\Gamma)}$ decays slowly

Recall that

$$\|F - P_n\|_{L^\infty(\Gamma)} \leq C\rho_*^{-n} \implies \|a^{(N)}\|_2 \lesssim C \cdot N.$$

Based on the assumption, for all $0 \leq n \leq N$,

$$\|F - P_n\|_{L^\infty(\Gamma)} \leq \rho_*^{N-n}\|F - P_N\|_{L^\infty(\Gamma)} = \rho_*^N\|F - P_N\|_{L^\infty(\Gamma)} \cdot \rho_*^{-n}.$$

Therefore,

$$\|a^{(N)}\|_2 \lesssim \rho_*^N\|F - P_N\|_{L^\infty(\Gamma)} \cdot N.$$

Implications: when $\|F - P_N\|_{L^\infty(\Gamma)}$ decays slowly

We've shown that $\|a^{(N)}\|_2 \lesssim N\rho_*^N \|F - P_N\|_{L^\infty(\Gamma)}$ in this case.

When $\|(V^{(N)})^{-1}\|_2 \lesssim \frac{1}{u}$, the monomial approximation error satisfies

$$\begin{aligned} \|F - \widehat{P}_N\|_{L^\infty(\Gamma)} &\lesssim \|F - P_N\|_{L^\infty(\Gamma)} + u \cdot \|a^{(N)}\|_2 \\ &\lesssim \|F - P_N\|_{L^\infty(\Gamma)} + u \cdot N\rho_*^N \|F - P_N\|_{L^\infty(\Gamma)} \\ &= (1 + u \cdot N\rho_*^N)\|F - P_N\|_{L^\infty(\Gamma)}. \end{aligned}$$

- When $N\rho_*^N \leq \frac{1}{u}$, we have that $\|F - \widehat{P}_N\|_{L^\infty(\Gamma)} \lesssim 2\|F - P_N\|_{L^\infty(\Gamma)}$.
- Recall that $\|(V^{(N)})^{-1}\|_2 \approx \rho_*^N$. So $N\rho_*^N \leq \frac{1}{u}$ generally holds when $\|(V^{(N)})^{-1}\|_2 \leq \frac{1}{u}$.

Implications: stagnation of convergence

We've shown that if $\|F - P_n\|_{L^\infty(\Gamma)}$

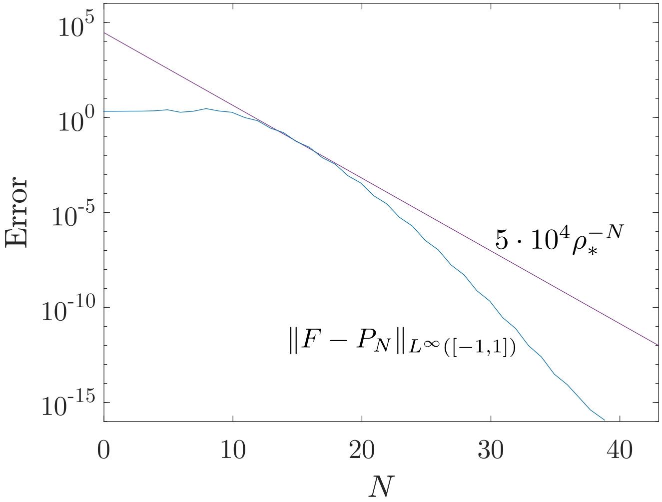
- decays at a rate **faster** than ρ_*^{-n},
- or decays at a rate **slower** than ρ_*^{-n},

then the monomial basis = a well-conditioned basis when the order $\leq$ threshold.

The only way for stagnation to happen before the order reaches the threshold is that, $\|F - P_n\|_{L^\infty(\Gamma)}$ first decays at a rate **slower** than ρ_*^{-n}, then starts to decay at a rate **faster** than ρ_*^{-n}.

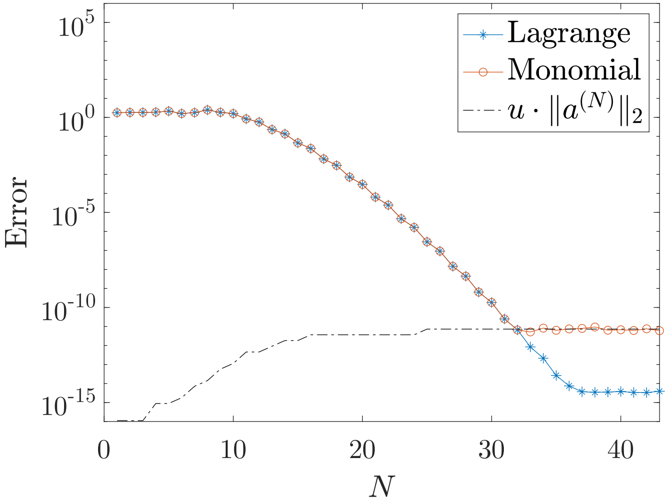
Examples: stagnation of convergence

$$F(x) = \cos(12x + 1)$$



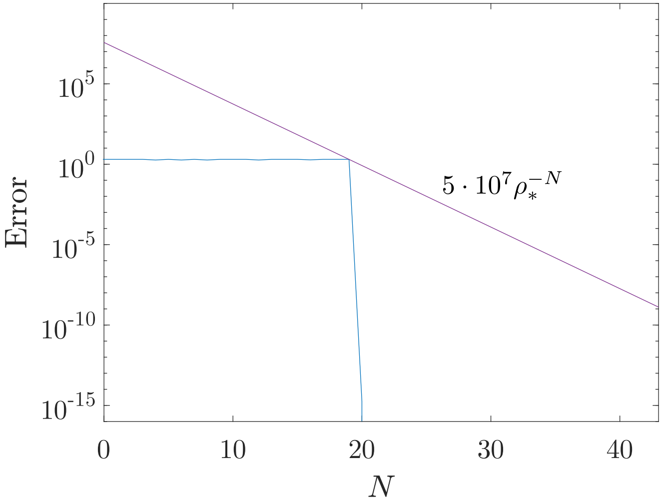
Examples

$$F(x) = \cos(12x + 1)$$



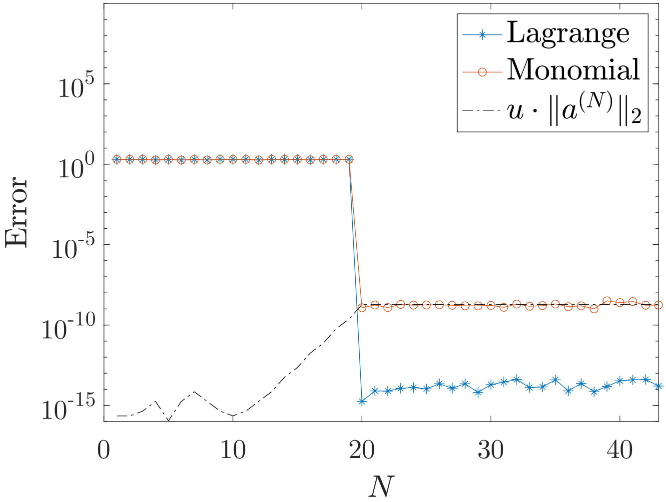
Examples

$$F(x) = T_{20}(x)$$



Examples

$$F(x) = T_{20}(x)$$



Implications: stagnation of convergence

- In practice, the interpolation error typically doesn't drop like crazy after decaying slowly.
- So stagnation of convergence typically only occurs when N is close to the threshold value.

How restrictive is the monomial basis?

- Extremely high-order interpolation is impossible due to the precondition $\|(V^{(N)})^{-1}\|_2 \lesssim \frac{1}{u}$.
- So **global** interpolation won't work.

How restrictive is the monomial basis?

On the other hand, **piecewise** polynomial interpolation in the monomial basis over a partition of Γ can be carried out stably, provided that

1. the maximum order of approximation over each subpanel is maintained below the threshold;
   Fine. The threshold isn't small and can be estimated easily.
2. the size of $u \cdot \|a^{(N)}\|_2$ is kept below the size of $\|F - P_N\|_{L^\infty(\Gamma)}$.
   1. Often satisfied automatically. If not, adding an extra level of subdivision almost always resolves the issue. Reducing the maximum order also helps.
   2. Even easier when high accuracy is not required.
   3. $u \cdot \|a^{(N)}\|_2$ can be easily estimated a posteriori.

The convergence rate of piecewise polynomial approximation is $\mathcal{O}(h^{N+1})$.

Rapid evaluations (short expansion, Estrin's scheme).

How restrictive is the monomial basis?

- The Vandermonde system is dense.
- Backward stable linear system solve generally takes $\mathcal{O}(N^3)$ operations.

Not a problem.

- The size of the Vandermonde matrix is not large ($\lesssim 50$ in 1-D).
- Highly optimized linear algebra libraries, e.g., LAPACK.
- When the domain is fixed, only need to factorize the matrix once.
- $\mathcal{O}(N^2)$ algorithms exist (could be less backward stable).

When $\Gamma \subset \mathbb{C}$ is a smooth simple arc

- One can approximate an analytic function $F : \Gamma \to \mathbb{C}$ by polynomials in the monomial basis.
- Key component in some of the layer potential evaluation algorithms (Helsing & Ojala 2008, af Klinteberg & Barnett 2021).
- The threshold ≈ 37 when Γ is a parabola.
- See our paper for experiments.

Generalization to higher dimensions

In 2-D, the Vandermonde matrix looks like

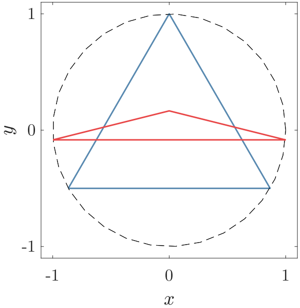
$$V^{(N)} := \begin{pmatrix} 1 & x_1 & y_1 & x_1^2 & x_1 y_1 & \cdots & y_1^N \\ 1 & x_2 & y_2 & x_2^2 & x_2 y_2 & \cdots & y_2^N \\ \vdots & \vdots & \vdots & \vdots & \vdots & \ddots & \vdots \\ 1 & x_{\widetilde{N}} & y_{\widetilde{N}} & x_{\widetilde{N}}^2 & x_{\widetilde{N}} y_{\widetilde{N}} & \cdots & y_{\widetilde{N}}^N \end{pmatrix},$$

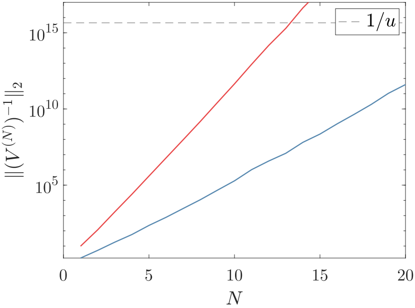
where $\widetilde{N}$ is the dimensionality of bivariate polynomials of order up to N.

Collocation points with relatively small Lebesgue constants have been constructed (Vioreanu & Rokhlin 2014).

The theory of monomial approximation is essentially same as 1-D.

Numerical experiments



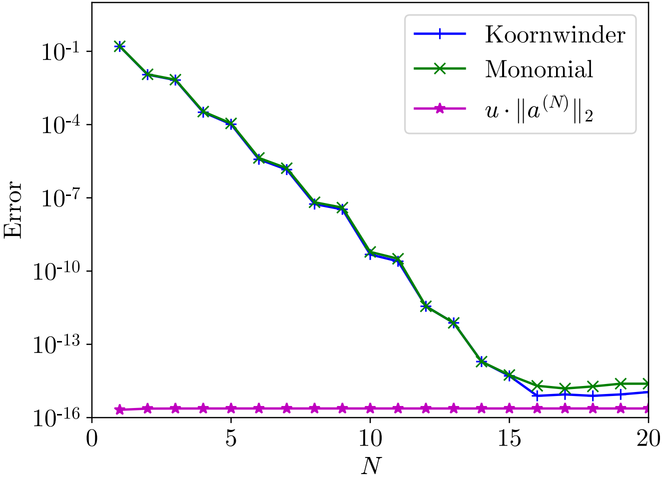


I'll show some experiments that compares the monomial basis with the Koornwinder polynomial basis over the blue triangle.

Correction (June 2023): we later found that bivariate polynomial interpolation in the monomial basis can be done stably up to 20th order on a triangle, **regardless of its aspect ratio**. See our paper "Rapid evaluation of Newtonian potentials over planar domains" for details.

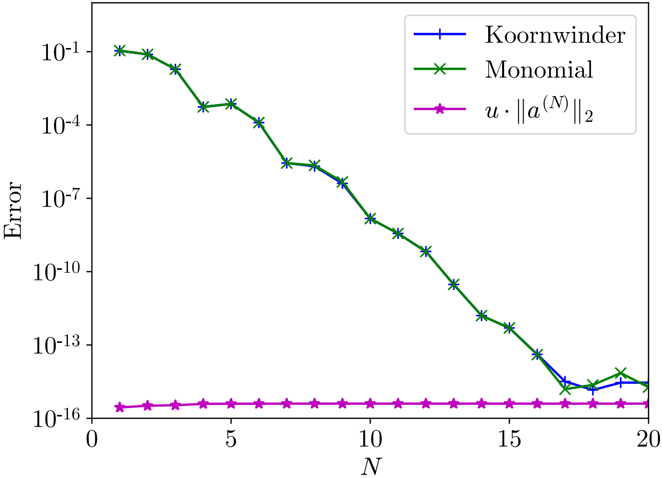
Numerical experiments

$$F(x, y) = e^{-(x^2+y^2)/4}$$



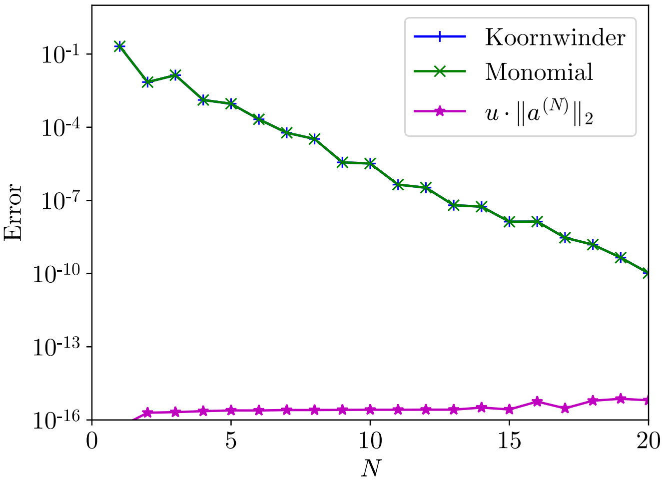
Numerical experiments

$$F(x,y) = \sin(xy/2 + x + y)$$



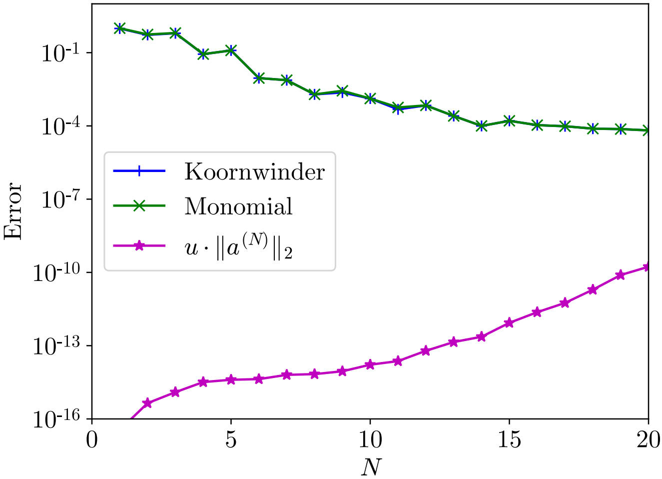
Numerical experiments

$$F(x,y) = \arctan(x) \cdot \arctan(y)$$



Numerical experiments

$$F(x, y) = |x + y|^{5.5}$$



Teaser: Newtonian potential evaluation

Given an irregular domain $\Omega \subset \mathbb{R}^2$ and a function $F : \Omega \to \mathbb{R}$, we're interested in calculating the Newtonian potential

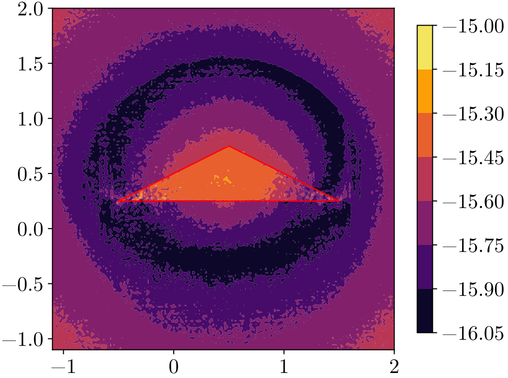
$$u(x) = \iint_\Omega \log(\|x - y\|) F(y) \, \mathrm{d}y.$$

- Triangulation.
- Approximate F over each mesh element using 2-D monomials.
- Compute the anti-Laplacian of the 2-D monomial expansion over each mesh element.
- Apply Green's third identity $\implies$ layer potentials over the boundaries of the mesh elements.

Z. Shen and K. Serkh. "*Rapid evaluation of Newtonian potentials on planar domains.*" arXiv:2208.10443 (2022).

Numerical experiments

$$F(x, y) = e^{-x^2-y^2}$$



Two orders of magnitude faster than adaptive integration.

Conclusions

Polynomial interpolation in the monomial basis is a valuable tool to have in the numerical toolbox.

Conclusions

An interactive demo:

https://uoft.me/monomial

Paper & slides are available on my personal website (`https://zewenshen.github.io`).

Bonus: what happens when the order > the threshold?

cos$(12x + 1)$, MATLAB's backslash